Part IIB 1995 Complex Algebraic Curves PMHW

Example Sheet I
(from Question 4 onwards, take $k = \mathbb{C}$)

1. Given points $P_0, \cdots, P_{n+1}$ in $\mathbb{P}^n = \mathbb{P}(W)$, no $(n+1)$ of which are contained in a hyperplane, show that homogeneous coordinates may be chosen on $\mathbb{P}(W)$ so that $P_0 = (1{:}\,0{:}\ldots{:}\,0)$, $\cdots$, $P_n = (0{:}\ldots{:}\,0{:}\,1)$ and $P_{n+1} = (1{:}\,1{:}\ldots{:}\,1)$.

2. Given hyperplanes $H_0, \cdots, H_n$ of $\mathbb{P}^n = \mathbb{P}(W)$ such that $H_0 \cap \ldots \cap H_n = \emptyset$, show that homogeneous coordinates $x_0, \cdots, x_n$ can be chosen on $\mathbb{P}(W)$ such that each H_i is defined be $x_i = 0$.

3. Show that the set of hyperplanes in $\mathbb{P}(W)$ is parametrized by $\mathbb{P}(W^*)$, where W^* is the dual vector space to W. If $P_1, \cdots, P_N$ are points of $\mathbb{P}(W)$, describe the set in $\mathbb{P}(W^*)$ corresponding to hyperplanes not containing any of the P_i. Deduce (assuming k infinite) that there are infinitely many such hyperplanes.

4. Let V be a hypersurface in $\mathbb{P}^n$ defined by a non-constant homogeneous polynomial F, and L a (projective) line in $\mathbb{P}^n$; show that V and L must meet.

5. Decompose that projective variety V in $\mathbb{P}^3$ defined by equations $X_2^2 = X_1X_3$, $X_0X_3^2 = X_2^3$ into irreducible components.

6. Show that the general equation for a projective conic $V \subset \mathbb{P}^2$ can be written in the form $\mathbf{x}'A\mathbf{x} = 0$, where A is a 3×3 symmetric matrix with entries in $\mathbb{C}$ and $\mathbf{x}' = (x_0{:}\,x_1{:}\,x_2)$; show that this equation is irreducible if and only if $\det(A) \neq 0$. If V is irreducible, show that V is isomorphic to $\mathbb{P}^1$.

7. Consider the projective plane curves corresponding to the following affine curves in $\mathbb{A}^2$.

   (a) $y = x^3$ (b) $xy = x^6 + y^6$
   (c) $x^3 = y^2 + x^4 + y^4$ (d) $x^2y + xy^2 = x^4 + y^4$
   (e) $2x^2y^2 = y^2 + x^2$ (f) $y^2 = f(x)$ with f a polynomial of degree n.

   In each case, calculate the points at infinity of these curves, and find the singular points of the projective curve.

8. If $V \subset \mathbb{P}^2$ is a projective plane curve defined by an irreducible homogeneous polynomial $F(X_0, X_1, X_2)$ of positive degree, show that the singular locus of V consists precisely of the points P in $\mathbb{P}^2$ with $\partial f/\partial X_i(P) = 0$ for $i = 0, 1, 2$.

9. Let $\phi\colon V \longrightarrow \mathbb{P}^m$ be a morphism from an irreducible projective variety V and assume that $W = \phi(V)$ is a subvariety of $\mathbb{P}^m$. Prove that W is irreducible.

10. Show that the projective plane curve with equation $X_0X_2^2 = X_1^3 + aX_0^2X_1 + bX_0^3$ is isomorphic to $V \subset \mathbb{P}^3$ defined by equations $X_0X_3 = X_1^2$, $X_2^2 = X_1X_3 + aX_0X_1 + bX_0^2$ via the map $\phi = (X_0^2{:}\,X_0X_1{:}\,X_0X_2{:}\,X_1^2)$

11. (Tripos Style Question)
    Explain briefly why a rational map $\phi\colon \mathrm{V} \dashrightarrow \mathbb{P}^m$ on a smooth projective curve is a morphism.

    Let V be the projective variety in $\mathbb{P}^3$ defined by $X_0X_3 = X_1X_2$ and L be the plane given by $X_0 = 0$. Let P be the point $(1{:}\,0{:}\,0{:}\,0)$. Show that the following recipe defines a rational map $\phi = (0{:}\,X_1{:}\,X_2{:}\,X_3)$ from V to L : For a point Q of V, the line through P and Q meets L in $\phi(Q)$. Show that ϕ is not a morphism.

    If V^* and L^* denote the intersections of V and L respectively with the plane $X_1 = X_2$, verify explicitly (i.e. not appealing to the above result) that ϕ induces an isomorphism between V^* and L^*.

Example Sheet II

(Throughout you may take $k = \mathbb{C}$. For $n = 3, 4, 5$, Question n uses Question (n-1).)

1. If P is a smooth point of an irreducible curve V and t a local parameter at P, show that $\dim_k \mathcal{O}_{V,P}/(t^n) = n$.

2. Suppose that $\phi : V \longrightarrow V$ is a surjective morphism of irreducible projective varieties for which the induced map on function fields $\phi^* = id_{k(V)}$; show that $\phi = id_V$.

3. Let $\phi : V \longrightarrow \mathbb{P}^1$ be a surjective morphism from an irreducible smooth projective curve V, and suppose that the induced map $\phi^* : k(\mathbb{P}^1) \to k(V)$ is an isomorphism. Demonstrate the existence of a morphism $\psi : \mathbb{P}^1 \to V$ with ψ^* the inverse of ϕ^*, and deduce that ϕ is an isomorphism.

4. Suppose that V is an irreducible smooth projective curve and P a point on V with $l(P) > 1$; show that V is isomorphic to $\mathbb{P}^1$.

5. For D any effective divisor on $\mathbb{P}^1$, show that $l(D) = \deg D + 1$. For D any non-zero effective divisor on V not isomorphic to $\mathbb{P}^1$, show that $l(D) \leqslant \deg D$.

6. Let P be the point at infinity on $\mathbb{P}^1$ and $D = 3P$; investigate the morphism ϕ_D. By choosing suitable subspaces of $\mathcal{L}(3P)$, obtain morphisms from $\mathbb{P}^1$ to $\mathbb{P}^2$ whose images are respectively the cuspidal cubic and the nodal cubic.

7. With notation as in Question 6, by considering a suitable subspace of $\mathcal{L}(4P)$, demonstrate the existence of a smooth curve $V \subset \mathbb{P}^3$ of degree 4 which is isomorphic to $\mathbb{P}^1$. (cf. Example Sheet I, Question 10 where we saw a smooth curve of degree 4 in $\mathbb{P}^3$ which is not isomorphic to $\mathbb{P}^1$.)

8. Assuming the fact that every smooth plane cubic in $\mathbb{P}^2$ has an inflexion point (i.e. a point $P \in V$ for which some line meets V at P with multiplicity $\geqslant 3$), show from first principles that homogeneous coordinates may be chosen on $\mathbb{P}^2$ with respect to which V has equation $X_0X_2^2 = X_1(X_1 - X_0)(X_1 - \lambda X_0)$ for some complex number $\lambda \neq 0, 1$.

9. Let P be the point at infinity of the plane curve with equation as in Question 8. Show that x/y is a local parameter at P, where $x = X_1/X_0$ and $y = X_2/X_0$. [Hint: Consider the affine piece $X_2 \neq 0$.] Hence calculate the numbers $v_P(x)$ and $v_P(y)$. Find a general form for a function in $\mathcal{L}(mP)$, and show that $l(mP) = m$ for $m > 0$.

10. An irreducible smooth projective curve V is covered by two affine pieces (with respect to different embeddings) which are affine plane curves with equations $y^2 = f(x)$ and $v^2 = g(u)$ respectively, where f is a square free polynomial of degree $2n$, and where $u = 1/x$ and $v = y/x^n$ in $k(V)$. Describe the polynomial $g(u)$ and show that the canonical divisor class of V has degree $2n - 4$.

11. (Tripos Style Question).

If f, $g \in k[x,y]$ are coprime polynomials over a field k, show that there exist polynomials α, $\beta \in k[x,y]$ such that $\alpha f + \beta g$ is a polynomial in x only.

If V and W are the affine plane curves defined by f and g respectively, and $P \in V \cap W$, we define the *intersection multiplicity* of the two curves of P to be $\dim_k \mathcal{O}_{\mathbb{A}^2,P}/(f,g)$. Show that this number is always finite, and calculate it in the case $f = y - x^2$, $g = y^2 - x^3$ and $P = (0,0)$.

12. (Tripos Style Question).

Let $y^{N-1} = f(x)$ be the equation of an affine plane curve $U \subset \mathbb{A}^2$, when f is a polynomial of degree N with distinct roots, and $V \subset \mathbb{P}^2$ be the corresponding projective plane curve (with equation $X_0 X_2^{N-1} = F(X_0, X_1)$). Prove that V is a smooth curve.

Let $P = (0:0:1)$; calculate $v_p(x)$ and $v_p(y)$. Deduce (without using Riemann-Roch) that $\mathcal{L}((n+1)P) = \mathcal{L}(nP) + 1$ for all $n > N(N+3)$.

13. (Paper IV Tripos Style Question).

Explain what is meant by the *genus* $g(V)$ and a canonical divisor K_V of a smooth projective curve V. Explain why $\deg(K_V)$ is a well-defined number and state the relation of this number to the genus.

If V is a smooth projective plane curve defined by an irreducible polynomial F of degree d, calculate both $g(V)$ and $\deg(K_V)$ from first principles and hence check that your stated relation holds in this case. [You may assume the fact that if U is an irreducible affine variety, any rational function which is everywhere regular on U will be an element of the coordinate ring $k[U]$].

Part IIB 1995 Complex Algebraic Curves PMHW

Example Sheet III

(Throughout you may assume the Riemann-Roch Theorem)

1. Let V be a smooth projective curve and P any point of V. Show there exists a non-constant rational function $f \in k(V)$ which is regular everywhere except at P. Show furthermore that there is a projective embedding of V for which the image of P is the only point at infinity.

2. Let V be a smooth projective curve of genus g; show that there is a non-constant morphism $\phi : V \to \mathbb{P}^1$ of degree $\leqslant g+1$.

3. Suppose that P_0 is a point on an elliptic curve V and $\phi_{3P_0} : V \to W \subset \mathbb{P}^2$ the corresponding embedding of V with image W. Show that $P \in W$ is an inflexion point if and only if $P \oplus P \oplus P$ is the identity under the group law on V determined by P_0. Deduce that if P, Q are inflexion points of W, so too is the 3rd point of intersection R of the line PQ with W

4. Let V be the plane cubic curve $zy^2 + z^2y = x^3 - xz^2$, and take the identity dement P_0 of the group law on V to be the point $(0:1:0)$ at infinity. If $P = (0:0:1)$, calculate the points $nP = P \oplus \cdots \oplus P$ of V for $n \leqslant 4$.

5. Let $\pi : V \to \mathbb{P}^1$ be a smooth hyperelliptic curve and $P \neq Q$ ramification points for π. Show that as elements of $Cl(V)$, $P - Q \neq 0$ but $2(P-Q) = 0$.

6. Let $\pi : V \to \mathbb{P}^1$ be a smooth hyperelliptic curve of genus $g > 1$ and Q any point of $\mathbb{P}^1$. We let ϕ^*Q denote the divisor of degree 2 on V corresponding to $Q \in \mathbb{P}^1$ i.e. $\phi^*Q = \sum_{\phi(P)=Q} v_p(\phi^*(t))P$, where t is a local parameter at Q. Show that $\mathcal{L}((g-1)\phi^*Q) \geqslant g$ on V, and hence deduce that $\mathcal{L}(K_V - (g-1)\phi^*Q) > 0$, i.e. that $K_V \sim (g-1)\phi^*Q$ on V. Use this to identify the space $\mathcal{L}(K_V)$, and also the image of the canonical map $\phi_{K_V} : V \to \mathbb{P}^{g-1}$.

7. Calculate the j-invariant of the Fermat cubic $x^3+y^3+z^3 = 0$. [HINT: Find linear combinations Y, Z of y, z such that $y^3 + z^3 = \frac{3}{4}ZY^2 + \frac{1}{4}Z^3$.]

8. Show that the plane cubic

$$x^3 + y^3 + z^3 - 3\lambda xyz = 0$$

fails to be smooth and irreducible if and only if $\lambda^3 = 1$. Adapt your argument from Question 7 to show that any elliptic curve may be embedded in $\mathbb{P}^2$ with an equation of the above form with $\lambda^3 \neq 1$.

9. If V is a smooth curve of genus 2 and D a divisor on V, show that ϕ_D is an embedding if and only if $\deg(D) \geqslant 5$.

10. If V is a smooth plane quartic, show that V is not hyperelliptic.

11. Let V be a smooth non-hyperelliptic curve of genus $g \geqslant 3$, and $\phi = \phi_{K_V} : V \to \mathbb{P}^{g-1}$ is canonical embedding.

Let H be a divisor on V corresponding to a hyperplane of $\mathbb{P}^{g-1}$; show that $\mathcal{L}(H) \geqslant g$ and hence that $\mathcal{L}(K_V - H) > 0$. Deduce that $H \sim K_V$, and also that any effective divisor $D \sim K_V$ corresponds to some hyperplane.

12. With the notation as in Question 11, suppose that $D = P_1 + \cdots + P_n$ is an effective divisor on V with the P_i distinct, and let Q_i denote the point $\phi(P_i)$ in $\mathbb{P}^{g-1}$ and Span $< Q_1, \ldots, Q_n >$ denote the linear subspace of $\mathbb{P}^{g-1}$ spanned by the points Q_i. Show that the Riemann-Roch Theorem for D can be interpretted geometrically as saying that $\mathcal{L}(D) = \deg D - \dim \mathrm{Span} < Q_1, \ldots, Q_n >$

13. (Tripos Style Question)

Are the following statements concerning complex projective curves true or false? Give a brief proof or a countexample as appropriate

(a) Any plane curve of degree $d > 3$ is non-rational

(b) Any smooth curve admits a morphism to $\mathbb{P}^1$

(c) Any smooth curve which admits a morphism from $\mathbb{P}^1$ will be isomorphic to $\mathbb{P}^1$

(d) Any morphism between a smooth curve of genus 4 and a smooth curve of genus 3 must be constant.

14**. Let $V \subset \mathbb{P}^2$ be a smooth cubic curve whose defining equation has real coefficients. Show that V has either 1 or 3 real inflexion points.

[HINT: Look up Sylvester's problem in an appropriate book.]