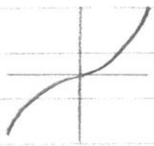
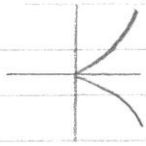
Complex Algebraic Curves

Let $\delta \in \mathbb{C}[X,Y]$ and consider zero locus, $\{(x,y) \in \mathbb{C}^2 : \delta(x,y) = 0\}$.

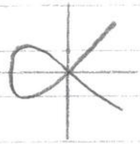
Eg. $y = x^3$:



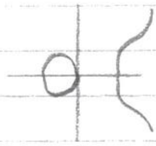
$y^2 = x^3$:



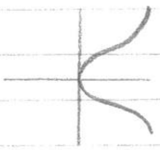
$y^2 = x^2(x+1)$:



$y^2 = x(x^2-1)$:



$y^2 = x(x^2+1)$:



- real loci, not considering complex.

Need to take coordinates in $\mathbb{C}$ and also consider behaviour at ∞.

Suppose W is a vector space over k of dimension $n+1$.
$\mathbb{P}(W) = \{$Lines through origin in $W\} = {}^{(W-\{0\})}/_{\sim}$: where $v \sim w$ iff $v = \lambda w$ for some $\lambda \in k^*$.
$\mathbb{P}(W)$ is a projective n-space, $\mathbb{P}^n$.

Definition: A linear subspace of dimension r in $\mathbb{P}(W)$ is a subset of the form $\mathbb{P}(V)$, with V a subspace of W of dimension $r+1$. It is called a hyperplane if $r = n-1$.

Note: $\mathbb{P}(U_1) \cap \mathbb{P}(U_2) = \mathbb{P}(U_1 \cap U_2)$. Moreover, if $\dim(\mathbb{P}(U_i)) = r_i$, then
$\cdot\ r_1 + r_2 \geq n \Rightarrow \dim U_1 + \dim U_2 \geq n+1 \Rightarrow \dim(U_1 \cap U_2) > 0 \Rightarrow \mathbb{P}(U_1) \cap \mathbb{P}(U_2) \neq \emptyset$.

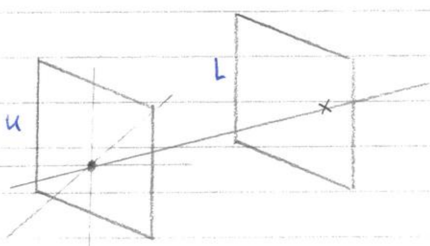
Eg. Two lines in $\mathbb{P}^2$ always meet.

Definition: An affine n-space, $\mathbb{A}^n$, over k is just an n-dimensional affine subspace of a vector space over k. Ie, a coset of an n-dimensional subspace.

If we choose a point, the affine n-space has the structure of a vector space.

Suppose $\mathbb{P}(U) \subset \mathbb{P}(W)$, hyperplane. Let L be any coset of U not containing the origin. L is an affine n-space. $\exists$ natural embedding of $L \hookrightarrow \mathbb{P}(W)$ with complement $\mathbb{P}(U)$.

Eg.



Point of L determines a unique point of $\mathbb{P}(W)$.

Conversely, if $L = U \times W$, any $v \in L$ can be written as $v = u + \lambda w$ ($u \in U$), uniquely.
Line $[v]$ does not meet L if $v \in U$, and meets L in a unique point if $v \notin U$.

Complement of a hyperplane in $\mathbb{P}^n$ is an affine n-space $\mathbb{A}^n$.

Homogeneous coordinates.

Choosing basis $e_0, \dots, e_n$ of W identifies W with k^{n+1} and then any point of $\mathbb{P}(W)$ is given by homogeneous coordinates, $\underline{x} = (x_0 : x_1 : \dots : x_n)$, (not all 0), where only the ratios matter. I.e. if $\lambda \in k^*$, $\lambda \underline{x}$ is the same point of $\mathbb{P}(W)$.

In terms of homogeneous coordinates, a linear subspace of $\mathbb{P}(W)$ is defined by homogeneous linear equations in these coordinates.

If $\mathbb{P}(U)$ is a hyperplane in $\mathbb{P}(W)$, choose $e_1, \dots, e_n$ basis for U, and extend to basis $e_0, \dots, e_n$ for W, then the hyperplane is given by $x_0 = 0$.
Complement of this hyperplane consists of classes $[v]$, where $v = (x_0 : x_1 : \dots : x_n)$, with $x_0 \neq 0$
Taking L to be the coset $e_0 + U$ (ie given by $x_0 = 1$), $\mathbb{P}(W) \setminus \mathbb{P}(U) \leftrightarrow L$, $(x_0 : \dots : x_n) \leftrightarrow (1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$.
Ie, have affine coordinates $(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$ on L. Ie, have identified L with k^n.
Conversely, given $(y_1, \dots, y_n) \in k^n$, have corresponding point $(1 : y_1 : \dots : y_n) \in \mathbb{P}(W) \setminus \mathbb{P}(U)$.

Eg. $y = x^3$, $y^2 = x^3$
$X_0^2 X_2 = X_1^3$ $\quad X_0 X_2^2 = X_1^3$.

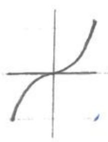
In this course, $k = \mathbb{C}$.

Definition: A projective variety $V \subset \mathbb{P}^n$ (with homogeneous coordinates $(x_0 : \dots : x_n)$) is defined to be the zero locus of a (finite) set of homogeneous polynomials in $k[x_0, \dots, x_n]$.

Remark: It is a fact (Hilbert's Basis Theorem) that any ideal in a polynomial ring is finitely generated, and so do not need to stipulate that the set is finite in the above definition.

$PGL(n+1, k) = GL(n+1, k)/k^*$.

Example: $\mathbb{P}^2$. Have affine coordinates $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$ on complement of line $X_0 = 0$ - we call this the "line at ∞." Sometimes we will write homogeneous coordinates as $(X ; Y ; Z)$. Then $Z = 0$ is usually thought of as the line at infinity and we have affine coordinates $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$ on complement.

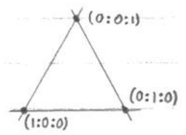
Eg. Recall the affine curves in $\mathbb{A}^2$: $y = x^3$:  $y^2 = x^3$: 

Corresponding projective curves given by: $X_0^2 X_2 = X_1^3$, $X_0 X_2^2 = X_1^3$

Definition: $V \subset \mathbb{P}^n$, projective variety. $I^h(V) :=$ ideal in $k[X_0, .., X_n]$ generated by homogeneous polynomials vanishing on V.

For $F \in k[\underline{X}]$, homogeneous, $F \in I^h(V)$ is shorthand for $F(\underline{x}) = 0 \ \forall \underline{x} = (x_0 : .. : x_n) \in V$.

A subvariety $V_1 \subset V$ is a subset defined by a finite number of further equations. Ie, V_1 is a subset and a projective variety.
Then, V is irreducible if it cannot be written as a union of proper subvarieties.

Eg. $X_0 X_1 X_2 = 0$ 

- 3 curves in $\mathbb{P}^2$; clearly reducible.

Lemma 1.1: V is irreducible $\Leftrightarrow$ for any homogeneous $F, G \in k[X]$ with $FG \in I^h(V)$ then either $F \in I^h(V)$ or $G \in I^h(V)$.

Proof: $(\Leftarrow)$ If $V = V_1 \cup V_2$, can choose homogeneous $F \in I^h(V_1) \setminus I^h(V)$, $G \in I^h(V_2) \setminus I^h(V)$.
$(\Rightarrow)$ If F, G homogeneous, $\notin I^h(V)$ with $FG \in I^h(V)$, then have proper subvarieties V_1, V_2 of V given by extra equation $F = 0$ (respectively $G = 0$) such that $V = V_1 \cup V_2$.

Rational functions on irreducible $V \subseteq \mathbb{P}^n$.

These are given by F/G with F, G homogeneous of the same degree and $G \notin I^h(V)$ such that equivalence relation $F/G \sim P/Q \Leftrightarrow FQ - GP \in I^h(V)$

A rational function f is regular at $p \in V$ if $\exists$ representation $f = F/G$ with $G(p) \neq 0$, and define value at p: $f(p) = \frac{F(p)}{G(p)}$.

Example: Projective plane curve with equation $X_0 X_2^2 = X_1^2 (X_1 + X_0)$. Rational function $f = \frac{X_2^2}{X_1(X_1 + X_0)}$ is regular at $p = (1:0:0)$ since $f = \frac{X_1}{X_0}$, with $f(p) = 0$.

Rational functions on V form a field in an obvious way. (Check. eg: $\frac{F}{G} + \frac{P}{Q} = \frac{FQ + PG}{GQ}$, etc).

Definition: An irreducible projective variety $V \subset \mathbb{P}^n$ is called a complex projective curve if $\mathbb{C}(V)$ is a finite extension $\mathbb{C}(t)$ for some $t \notin \mathbb{C}$.

In this case, $\mathbb{C}(V)$ is a finite extension of $\mathbb{C}(s)$ for any non-constant $s \in \mathbb{C}(V)$

$\ulcorner$ s satisfies some equation with coefficients in $\mathbb{C}(t) \Rightarrow \exists$ irreducible polynomial in two variables such that $f(s,t) = 0$ in $\mathbb{C}(V)$. ($s \notin \mathbb{C} \Rightarrow f$ involves t), $\Rightarrow t$ satisfies some equation with coefficients in $\mathbb{C}(s)$. Ie, $\mathbb{C}(t,s)/\mathbb{C}(s)$, finite extension. But $\mathbb{C}(V)$ is finite over $\mathbb{C}(t)$ and hence over $\mathbb{C}(t,s)$. Hence $\mathbb{C}(V)/\mathbb{C}(s)$ finite. $\lrcorner$

Definition: An affine variety $V \subset \mathbb{A}^n$ is defined to be the zero locus of a finite set of polynomials in $k[Y_1, .., Y_n]$

We now define the function field $k(V)$ in an analogous way to the projective definition. For V irreducible, argument of lemma 1.1 $\Rightarrow$ ideal $I(V)$ of polynomials vanishing on V is prime.
A rational function h on V is given as a quotient $\frac{f}{g}$, $g \notin I(V)$, subject to obvious equivalence relation. [Then the field of fractions of the integral domain, $k[Y_1,..,Y_n]/I(V) = k[V]$, the ring of polynomial functions on V – the coordinate ring].

If now $V \subset \mathbb{P}$, irreducible projective variety, and $V \not\subset \{X_0 = 0\}$, then $V_0 \subset \mathbb{A}^n_0$, affine piece given by $X_0 = 0$. In $\mathbb{A}^n$ have affine coordinates $y_i = \frac{X_i}{X_0}$, $i = 1,..,n$.
Clearly, if $F \in I^h(V)$ then $F(1, y_1, .., y_n) \in I(V_0)$. Observe conversely, if $f \in I(V_0)$, polynomial of degree d, its homogenisation $F = f^h = X_0^d f(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}) \in I^h(V)$.
[Since F vanishes on $V_0 = V \cap \mathbb{A}^n_0$, so $X_0 F \in I^h(V) \Rightarrow F \in I^h(V)$, by lemma 1.1].
Hence see, from Lemma 1.1, that $I(V_0)$ prime $\Rightarrow V_0$ irreducible and $\exists$ isomorphism $k(V) \to k(V_0)$, given in obvious way - "put $X_0 = 1$."

$$\left[\frac{F(X_0,..X_n)}{G(X_0,..,X_n)}\right] \longmapsto \left[\frac{F(1,y_1,..,y_n)}{G(1,y_1,..,y_n)}\right]$$, noting that if G vanishes on V_0, then $G \in I^h(V)$.

Affine Cover.

For $V \subset \mathbb{P}^n$ (not contained in hyperplane), also have affine pieces $V_i \subset \mathbb{A}^n_i$ given by $X_i \neq 0$. [Affine coordinates: $(\frac{X_0}{X_i}, .. \widehat{\frac{X_i}{X_i}}, \dots \frac{X_n}{X_i})$, where '^' indicates omit], and $k(V)$ is determined by any of these affine pieces. Moreover, $V = \bigcup_{i=0}^{n} V_i$ and so much of the detailed geometry of V may be studied by looking at the affine variations V_i - the affine cover.

Eg: curve $V \subset \mathbb{P}^2$ with equation $X_0 X_2^2 = X_1^3$. Have affine pieces: $V_0 : y^2 = x^3$, $V_1 : uv^2 = 1$, $V_2 : w = z^3$.
cf. Riemann sphere, $\mathbb{P}^1(\mathbb{C})$, homogeneous coordinates $(x_0 : x_1)$, $= \mathbb{C} \cup \{\infty\}$, where $\infty = (0:1)$.
On $\mathbb{P}^1 \setminus \{\infty\}$, $= \mathbb{A}^1$, take affine coordinate $z = \frac{x_1}{x_0}$, on $\mathbb{P}^1 \setminus \{0\} = \mathbb{A}^1$, take affine coordinate $\frac{1}{z} = \frac{x_0}{x_1}$.

Lemma 1.2: Given coprime $f, g \in k[x,y]$, $\exists\ \alpha, \beta \in k[x,y]$ such that $\alpha f + \beta g = h$, where $0 \neq h \in k[x]$.
Proof: We write $f = a_m(x) y^m + .. + a_1(x) y + a_0(x)$ and $g = b_n(x) y^n + .. + b_0(x)$, wlog $n \geq m$.
Set $g_1 = b_n y^{n-m} f - a_m g$, a polynomial of degree $\leq n-1$ in y.
Now, recall $k[x,y]$ is a UFD. If $g_1 = 0$, then any irreducible factor of f divides $a_m(x) \Rightarrow f \in k[x]$, so done.
If $g_1 \neq 0$, then any common factor of f and g_1 divides $a_m g$ and hence is a polynomial in x only. So $f = \gamma \tilde{f}$, $g_1 = \gamma \tilde{g}_1$ for some $\gamma \in k[x]$, where $\tilde{f}$ and $\tilde{g}_1$ are coprime. But induction on the degree of y, $\exists\ \alpha_1, \beta_1$ such that $\alpha_1 \tilde{f} + \beta_1 \tilde{g}_1 \in k[x]$,
$\Rightarrow \alpha_1 f + \beta_1 g_1 \in k[x] \Rightarrow \exists\ \alpha, \beta$ such that $0 \neq \alpha f + \beta g \in k[x]$.

Now consider $V \subset \mathbb{P}^2$ defined by homogeneous polynomial $F(X_0, X_1, X_2)$ of positive degree, where $X_0 \nmid F$. Observe: F irreducible $\Leftrightarrow$ $f(x,y) = F(1,x,y)$ irreducible.

Corollary 1.3: If F irreducible, then the only proper subvarieties of V in $\mathbb{P}^2$ consist of finite sets of points. Since V itself consists of infinitely many points, V must be irreducible. (ie., not a finite union of subvarieties).

Proof: A proper subvariety is given by vanishing of homogeneous polynomials $G \notin I^h(V)$. Suppose $G \notin I^h(V)$ and let $f, g \in k[x,y]$ be polynomials corresponding to $F, G \in k[X_0, X_1, X_2]$. (wlog, V not line $X_0 = 0$). Since F irreducible, it is easily seen that curve V has only finitely many points on $\{X_0 = 0\}$, "line at ∞." [Put $X_0 = 0$ in F to get homogeneous polynomial in X_1, X_2 - only finitely many zeroes $(x_1 : x_2)$].
Thus need to prove $\{f=0\} \cap \{g=0\}$ finite and this follows from lemma 1.2 since $\exists$ only finitely many x-coordinates and finitely many y-coordinates.
Last part (V infinite) is easy to check, since except for finitely many values $x = a$, say, we can solve $f(a,y)$ for value in y.
So, $V \neq V_1 \cup V_2$ for V_1, V_2 proper subvarieties.

In this case, consider $k(V)$. It is naturally isomorphic to $k(V_0)$, where $V_0 = V \cap \{x_0 \neq 0\} \subset \mathbb{A}^n_0$, which in turn is just the field of fractions of $k[X,Y]/I(V_0)$.
But $g \in I(V_0) \Rightarrow \{f=0\} \cap \{g=0\}$ infinite $\Rightarrow f, g$ not coprime, ie. $f | g$. Thus, $I(V_0) = (f)$, and $k(V) \cong$ field of fractions of $k[X,Y]/(f)$. Assuming, wlog, that f does not involve y, $\mathbb{C}(V)$ is then a finite extension of $\mathbb{C}(x)$ and so V is indeed a curve in sense of previous definition. V is called a plane projective curve.

Local ring of an irreducible variety V at a point P.

Let V be an irreducible (projective) variety, $P \in V$. $\mathcal{O}_{V,P} := \{h \in k(V) \mid h \text{ regular at } P\}$.
This is a subring of $k(V)$ with maximal ideal $m_{V,P} := \{h \in \mathcal{O}_{V,P} \mid h(P) = 0\}$. [Have singular evaluation map $\mathcal{O}_{V,P} \to k$ given by $h \mapsto h(P)$ whose kernel is $m_{V,P}$, and so $\mathcal{O}_{V,P} / m_{V,P} \cong k$]
Units (elements invertible in $\mathcal{O}_{V,P}$), $U(\mathcal{O}_{V,P}) = \mathcal{O}_{V,P} \setminus m_{V,P}$. Ie. $m_{V,P} =$ non-units of $\mathcal{O}_{V,P}$.
Deduce $m_{V,P}$ is unique maximal ideal of $\mathcal{O}_{V,P}$. Ie, $\mathcal{O}_{V,P}$ is a local ring (definition).

Suppose $V \in \mathbb{P}^n$, and V_0 affine piece given by $X_0 \neq 0$. Under the isomorphism $k(V) \xrightarrow{\cong} k(V_0)$, $\mathcal{O}_{V,P}$ corresponds to the subring of $k(V_0)$ consisting of quotients f/g with $f, g \in k[V_0]$, $g(P) \neq 0$. So if $P \in V_0$, Local ring $\mathcal{O}_{V,P}$ determined by affine piece V_0.

Definition: A local ring A with maximal ideal m is called a discrete valuation ring, (DVR), if $\exists$ $t \in m$ such that every non-zero element $a \in A$ can be ~~a~~ written in the form $a = ut^n$ for some $n \geq 0$ and a unit u.
t is called a local coordinate or local parameter at P.

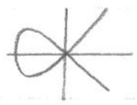
Easy to check that the factorisation is unique.

Eg. $V = \mathbb{A}^1$, $P = 0 \in \mathbb{A}^1$. Then $\mathcal{O}_{V,P} = \{\frac{f}{g}$ with f, g coprime and $g(0) \neq 0\}$. Clear that any element (non-zero) of $\mathcal{O}_{V,P}$ can be written as ux^n for $n \geq 0$ and some unit u.

For V an irreducible algebraic curve, say that $P \in V$ is a smooth or non-singular point if $\mathcal{O}_{V,P}$ is a DVR.

Lemma 1.4: An affine ~~plane~~ curve $U \subset \mathbb{A}^2$ given by irreducible $f \in k[x,y]$ is singular at P iff $\frac{\partial f}{\partial x}(P) = 0 = \frac{\partial f}{\partial y}(P)$.

Proof: Wlog, $P = (0,0)$. Write $f = f_1 + f_2 + \dots$, $\deg f_i = i$. Then lemma asserts that $f_1 = 0 \iff P$ is singular. Suppose P is non-singular. $\exists$ local parameter $t \in \mathcal{O}_{U,P}$ (wlog $t \in k[U]$) such that $x = u_1 t^r$, $y = u_2 t^s$ (u_1, u_2 units). Since $m_{V,P} = (x,y)$, one of r and s must be 1. Wlog $s = 1$. $\therefore x = uy^r$ for some unit $u \in \mathcal{O}_{V,P}$.
Write $u = \frac{v_1}{v_2}$, with $v_i \in k[x,y]$, $v_i(P) \neq 0$ $[i = 1,2]$. $\therefore v_2 x = v_1 y^r$ as elements of $k[V]$. Therefore, polynomial $v_2 x - v_1 y^r \in I(V) = (f) \Rightarrow f \mid (v_2 x - v_1 y^r) \Rightarrow f$ has a linear term $\Rightarrow f_1 \neq 0$.
Conversely, suppose $f_1 \neq 0$, and affine coordinates have been chosen such that $P = (0,0)$ and $f = x - y +$ (higher order terms). Thus $f = xp(x) - yq(x,y)$ with $p(0) \neq 0$, $q(0,0) \neq 0$. In particular, $x = vy$ in $\mathcal{O}_{V,P}$ with $v = \frac{q}{p} \in \mathcal{O}_{V,P}$, a unit.
Claim: $\mathcal{O}_{V,P}$ is a DVR with local parameter y. Proof: See printed notes.

Eg: $y^2 = x^2(x+1)$  Singular at $(0,0)$, smooth elsewhere.
$\frac{\partial}{\partial x} = x(3x+2)$, $\frac{\partial}{\partial y} = 2y$, $y^2 = x^2(x+1)$

Only possible solution is $(0,0)$. Corresponding projective curve in $\mathbb{P}^2(\mathbb{C})$ has equation $X_0 X_2^2 = (X_1 + X_0) X_1^2$, with unique point "at ∞", $(0:0:1) = Q$.
Now consider affine piece given by $X_2 \neq 0$ and get affine equation $u = v^2(u+v)$, with $Q = (0,0)$ and clearly smooth at Q.

cf. Exercise 8 on sheet 1: $V \subset \mathbb{P}^2$ given by homogeneous equation $F(X_0, X_1, X_2) = 0$, then the singular points of V are just $P \in \mathbb{P}^2$ such that $\frac{\partial F}{\partial X_i}(P) = 0$ $\forall i$.

For $\mathcal{O}_{V,P}$, DVR, we have a function $v_P : k(V)^* \to \mathbb{Z}$, (valuation at P), where $v_P(ut^n) = n$.
(Independent of choice of local parameter t, u a unit in $\mathcal{O}_{V,P}$)
Note: $v_P(f+g) \geq \min(v_P(f), v_P(g))$. $v_P(fg) = v_P(f) + v_P(g)$.

V an irreducible projective variety. A rational map $\varphi : V \dashrightarrow \mathbb{P}^m$, (where $\dashrightarrow$ means that φ is not a function on the whole of V, ie. $\exists$ poles, etc.), is given by an $(m+1)$-tuple $(f_0 : f_1 : \dots : f_m)$ of elements of $k(V)$, (not all 0) modulo relation that $(f_0 : \dots : f_m)$ and $(h_0 : \dots : h_m)$ define the same rational map if $\exists\, h \in k(V)^*$ such that $h_i = h f_i$ $\forall i$.

Interpreting the f_i given by $f_i = \frac{H_i}{G_i}$ (H_i, G_i - homogeneous polynomials of same degree, $G_i \notin I^h(V)$), and clearing denominators, can represent rational function by an $(m+1)$-tuple $(F_0 : \dots : F_m)$ of homogeneous polynomials of the same degree with at least one $F_i \notin I^h(V)$, where $(F_0 : \dots : F_m)$ and $(G_0 : \dots : G_m)$ define the same rational function iff $F_i G_j - F_j G_i \in I^h(V)$ $\forall i,j$.

Say φ is regular at $P \in V$ if it can be written as $(f_0 : .. : f_m)$ with $f_i \in \mathcal{O}_{V,P}$ and at least one not vanishing at P, and then $\varphi(P) = (f_0(P) : .. : f_m(P))$ $\Leftrightarrow$ $\exists$ representation $(F_0 : .. : F_m)$ with $F_i(P) = 0$ for some i, and then $\varphi(P) = (F_0(P) : .. : F_m(P))$.
$W \subset \mathbb{P}^m$, $\varphi : V \dashrightarrow W$ means that $\varphi(P) \in W$ whenever defined.

A morphism $\varphi : V \to W$ is an everywhere regular rational map.
An isomorphism $\varphi : V \to W$ is a morphism with an inverse morphism $\psi : W \to V$.
Given a morphism (or even a rational map) $\varphi : V \dashrightarrow W$ of irreducible projective varieties and $f \in k(W)$, we can define $\varphi^*(f) = f \circ \varphi \in k(V)$ in obvious way, provided $\varphi(V)$ is not contained in a proper subvariety of W. [If $\varphi = (h_0 : .. : h_m)$ and $f = \frac{F(Y_0, .., Y_m)}{G(Y_0, .., Y_m)}$, $G \notin I^h(W)$, define $\varphi^*(f) = \frac{F(h_0, .., h_m)}{G(h_0, .., h_m)} \in k(V)$, easily verified to be well-defined.

Definition: $\varphi : V \dashrightarrow W$ as above is birational if $\varphi^* : k(W) \to k(V)$ is an isomorphism.

Clearly, if φ is an isomorphism with inverse $\psi : W \to V$, then have corresponding inverse morphism $\psi^* : k(V) \to k(W)$. [Follows since $(\varphi\psi)^* = \psi^*\varphi^*$ and $(\psi\varphi)^* = \varphi^*\psi^*$].

Eg: $\varphi : \mathbb{P}^1 \to V \subset \mathbb{P}^2$. V given by $X_0X_2^2 = X_3^2$. φ is given by triple $(s^3 : st^2 : t^3)$, induces an isomorphism on function field. [Look affinely, since V_0 is given by $y^2 = x^3$ and $k[V_0] = k[x,y]/(y^2 - x^3) \cong k[t^2, t^3]$ and note $k(V_0) \cong k(t)$].

If now $\varphi(P) = Q$, we can choose a representation $(h_0 : .. : h_m)$ ~~such that~~ for φ (φ regular at P), such that h_i regular at Q. Write $f = \frac{F}{G}$, with $G(Q) \neq 0$, and then $\varphi^*(f) = \frac{F(h_0, .., h_m)}{G(h_0, .., h_m)}$ is regular at P.

Ie.
$$\begin{array}{ccc} \varphi^* : \mathcal{O}_{W,Q} & \hookrightarrow & \mathcal{O}_{V,P} \\ \cap & & \cap \\ k(W) & \longleftrightarrow & k(V) \end{array}$$

In particular, if φ is an isomorphism, then $\varphi^* : \mathcal{O}_{W,\varphi(P)} \xrightarrow{\cong} \mathcal{O}_{V,P}$

Example: Consider morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^3$; $(s,t) \mapsto (s^3 : s^2t : st^2 : t^3)$. Image is contained in variety V with equations $X_1X_2 = X_0X_3$, $X_1^2 = X_0X_2$, $X_2^2 = X_1X_3$.
Conversely, given $(x_0 : x_1 : x_2 : x_3) \in V$, note either $x_0 \neq 0$ or $x_3 \neq 0$. Say wlog $x_0 \neq 0$. Choose $s \in \mathbb{C}$ such that $x_0 = s^3$ and $t \in \mathbb{C}$ such that $x_1 = s^2t$.
Equations $\Rightarrow$ $x_2 = st^2$, $x_3 = t^3$. $\therefore$ $\varphi : \mathbb{P}^1 \to V$, say V is called the twisted cubic in $\mathbb{P}^3$.
Since the only proper subvarieties of V are finite sets of points, [if $F(X_0, X_1, X_2, X_3) \notin I^h(V)$, then $F(s^3, s^2t, st^2, t^3)$ is a non-zero homogeneous polynomial in $k[s,t]$ with only finitely many solutions in $(s:t)$], and hence V is irreducible.
Moreover, φ is an isomorphism with inverse $\psi = (X_0, X_1) = (X_2, X_3)$.
In particular, twisted cubic is smooth.

Proposition 1.6: If V a curve and $P \in V$ a smooth point, $\varphi : V \dashrightarrow \mathbb{P}^m$ rational map, then φ regular at P. In particular, if V is smooth then φ is a morphism.

Proof: $\mathcal{O}_{V,P}$ is a DVR, local parameter t. If $\varphi = (h_0 : .. : h_m)$ with non-zero h_i of the form $h_i = u_i t^{n_i}$ (u_i - unit, $n_i \in \mathbb{Z}$), clearing denominators and cancelling powers of t, can assume $n_i \geq 0$ $\forall i$ and $n_j = 0$, some j. $\Rightarrow$ φ regular at P.

Proposition 1.5: Non-zero, smooth $f \in \mathbb{C}(V)$ has finitely many zeroes and poles.

Proof: Wlog can take affine smooth curve $U \subset \mathbb{A}^n$ and $f \in k(U)$. Choose coordinate function x, non-constant on U, and let $\pi : U \to \mathbb{A}^1$ be corresponding projection - this corresponds to inclusion of fields $k(x_1) \hookrightarrow k(U)$ which is finite.

Claim 1: For each value of x, $\exists$ only finitely many values $Q \in U$. For each x_i $(i>1)$, satisfies an equation $f_i(x, x_i) = 0$. $c_m(x_1)x_i^m + \dots + c_1(x_1)x_i + c_0(x_1) = 0$, ($c_j(x_1)$ have no common factor).

Claim 2: f has only finitely many poles on U. f satisfies an equation $b_r(x_1)f^r + \dots + b_1(x_1)f + b_0(x_1) = 0$, with $b_i(x_1)$ polynomials in x_1. If $P = (a_1, \dots, a_n)$ is a pole of f, then $v_P(b_r) > 0$. Claim 1 $\Rightarrow$ Claim 2. Applying this to $\frac{1}{f}$, have only finitely many zeroes.

Corollary 1: Hypothesis as in proposition 1.5. A proper subvariety W of V is finite.

Proof: If $F \notin I^h(V)$, homogeneous of degree m, rational functions x_i^m/F have finitely many poles for each i $\Rightarrow$ F has finitely many zeroes on V.

Corollary 2: If $\varphi : V \to W$ is a non-constant morphism between smooth projective curves, then φ has finite fibre. (ie. $\varphi^{-1}(Q)$ finite $\forall\, Q \in W$).

Proof: Suppose $Q \in W$ and $\pi : W \to \mathbb{P}^1$ a projective morphism with composite $\psi = \pi\varphi : V \to \mathbb{P}^1$ non-constant and wlog $\pi(Q) = (1:0)$. Let x be a local parameter at $0 \in \mathbb{A}^1$ (x affine coordinate), then $\frac{1}{x}$ has a pole only at $(1:0)$ at $\mathbb{P}^1$. Then $h = \psi^*(\frac{1}{x})$ has only finitely many ~~points~~ poles (by proposition 1.5). However, any $P \in V$ with $\varphi(P) = Q$ is a zero of $\psi^*(x)$ and hence a pole of $\psi^*(\frac{1}{x})$. $\therefore \varphi$ has finite fibres.

In the above situation, image of φ is not a finite set of points. Hence $\exists$ induced injective homomorphism $\varphi^* : k(W) \hookrightarrow k(V)$. The degree, $\deg \varphi := [k(V) : \varphi^* k(W)]$.

Finiteness Theorem: $\varphi : V \to W$, non-constant morphism between smooth projective curves and $Q \in W$ and ~~to~~ t local parameter at Q, then $\sum_{P \in \varphi^{-1}(Q)} v_P(\varphi^*(t)) = \deg \varphi$.

2. Divisors.

Rational maps (and therefore morphisms, if V smooth) on a curve V are given by $(m+1)$-tuples of rational functions $\varphi : V \dashrightarrow \mathbb{P}^m$, $\varphi(V) \not\subset$ hyperplane $\Leftrightarrow$ rational functions are linearly independent over k.

Let $L \subset k(V)$ be the $(m+1)$-dimensional k-vector space spanned by $f_0, \dots, f_m$. ($\varphi = (f_0 : \dots : f_m)$). Then up to an action of $PGL(n+1, k)$ $[= GL(n+1, k)/k^*]$, the rational map φ is determined by the vector space L.

If $h \in k(V)^*$, the vector space hL defines the same rational map.

Divisors give a tool for picking out certain finite dimensional subspaces of $k(V)$ and hence rational maps, by limiting possible poles.

Examples: $V = \mathbb{P}^1$, $(x_0 : x_1)$, $P = \infty$.

Rational functions are of the form $\dfrac{\prod_{i=1}^{m}(\beta_i x_1 - \alpha_i x_0)}{\prod_{i=1}^{m}(\delta_i x_1 - \gamma_i x_0)}$. (Wlog $\nexists$ common factors)

Everywhere regular rational functions on V are just constants k ($\nexists$ poles $(\gamma_i : \delta_i)$).

So allow pole at $\infty = (0:1)$. If h is regular on $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{P\}$, then $h = \prod_{i=1}^{m}(\beta_i x - \alpha_i)$, where $x = \frac{x_1}{x_0}$ affine coordinate on $\mathbb{A}^1$, ie. h a polynomial of degree m in $k[x]$. As $\frac{1}{x} = \frac{x_0}{x_1}$ is a local parameter at $P = \infty = (0:1)$, we see $v_P(h) = -m$.

Let $\mathcal{L}(mP)$ denote the subspace of $k(V)$ given by $h \in k(V)^*$, regular on $V \setminus P$, with $v_P(h) \geq -m$, (together with 0). - viz. polynomials $h \in k[x]$ of degree $\leq m$.

Writing down a basis we obtain morphism $\varphi_{mP} : \mathbb{P}^1 \to \mathbb{P}^m$.

Eg. $m = 3$. $\mathcal{L}(3P)$ has basis: $1, x, x^2, x^3$ and so $\varphi_{3P} = (1 : x : x^2 : x^3) = (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3)$ is just the isomorphism of $\mathbb{P}^1$ onto the twisted cubic described before.

If $Q = (1:0) = 0 \in \mathbb{A}^1$, let $\mathcal{L}(2P+Q) = \{0\} \cup \{h \in k(V)^* : h \text{ regular on } V \setminus \{P, Q\} \text{ with } v_P(h) \geq -2, v_Q(h) \geq -1\}$.

We have a corresponding basis: $x^{-1}, 1, x, x^2$ for this space, and so we get the same rational map as before, $(x^{-1} : 1 : x : x^2)$.

Let $\mathcal{L}(4P - Q) = \{0\} \cup \{h$, regular function of $V \setminus \{P\}$, with $v_P(h) \geq -4$, $v_Q(h) \geq 1$ - ie. zero at $Q\}$.

Have corresponding basis x, x^2, x^3, x^4 and so get the same rational map φ.

Definition: V a smooth projective curve. A divisor D on V is a formal finite sum $D = \sum n_i P_i$, with $P_i \in V$, $n_i \in \mathbb{Z}$.

They form a free abelian group, $\mathrm{Div}(V)$, under addition. The degree, $\deg D = \sum n_i$.

In section 1, we saw that $f \in k(V)^*$ has only finitely many poles and zeroes. Let $(f) := \sum_{P \in V} v_P(f) P$, (formal sum), the principal divisor associated to f. [Note that $v_P(f) = 0$ except for finitely many points]

Observe: $(fg) = (f) + (g)$, ie $v_P(fg) = v_P(f) + v_P(g)$.

If $f \in k^*$, constant, then $(f) = 0$.

If $f \notin k^*$, consider $\varphi = (1 : f) : V \to \mathbb{P}^1$. Zeroes of f are just $\varphi^{-1}(1:0)$, poles are $\varphi^{-1}(0:1)$. $x = \frac{x_1}{x_0}$ is an affine coordinate on $\mathbb{A}^1 = \{x_0 \neq 0\}$, and a local parameter at $0 = (1:0) \in \mathbb{P}^1$, and $\frac{1}{x} = \frac{x_0}{x_1}$ - a local parameter at $\infty = (0:1)$.

Now, observe: $\varphi^*(x) = f$. $(f) = \sum_{P \in \varphi^{-1}(0)} v_P(\varphi^*(x)) - \sum_{P \in \varphi^{-1}(\infty)} v_P(\varphi^*(\frac{1}{x}))$.

Finiteness Theorem for $\varphi \Rightarrow (f) \neq 0$, but $\deg f = \deg \varphi - \deg \varphi = 0$.

Ie principal divisors have degree 0.

Two divisors D_1, D_2 are linearly equivalent, $D_1 \sim D_2$, if $D_1 - D_2$ is principal. Linear equivalence classes form a group under addition, the divisor class group:

$Cl(V) := \mathrm{Div}(V) / (\text{subgroup of principal divisors})$.

Example: For $V = \mathbb{P}^1$, see easily that D is principal $\Leftrightarrow \deg D = 0$, and so $Cl(V) \cong \mathbb{Z}$.

In general, we have a homomorphism of abelian groups, deg: $Cl(V) \to \mathbb{Z}$.

Definition: D is effective, $D \geq 0$ if $n_i \geq 0 \ \forall i$

For $V \subset \mathbb{P}^n$ a smooth projective curve, $F \in k[X_1, \dots, X_n]$ homogeneous of degree d and $F \notin I^h(V)$ we can define a divisor (F) on V.
If $P \in V_i := V \cap \{X_i \neq 0\}$ we can define $v_P(F) := v_P\left(\frac{F}{X_i^d}\right)$, clearly well-defined since if $P \in V_j$, then $v_P\left(\frac{X_i}{X_j}\right) = 0$. Ie, on V_i, we'll define divisor to coincide with (F/X_i^d). Define $(F) = \sum_{P \in V} v_P(F) P$.
Clearly well-defined divisor, called the divisor on V "cut out by the hyperplane $\{F=0\}$" and effective.

If F, G as above, homogeneous of same degree, then $(F) = (G) + (F/G)$. Ie $(F) \sim (G)$.

Definition: The degree of smooth projective curve $V \subset \mathbb{P}^n$ is the degree of any divisor (L) cut out by hyperplane $L = 0$ where $L \notin I^h(V)$. [L homogeneous linear form]. - well-defined, since linear equivalent divisors have same degree.

Example: Twisted cubic $V \subset \mathbb{P}^3$. $\varphi: \mathbb{P}^1 \to V$; $(s,t) \mapsto (s^3 : s^2t : st^2 : t^3)$
For $L \in k[X_0, X_1, X_2, X_3]$, homogeneous linear form, $\deg(L)$ on V is just the degree of the divisor on $\mathbb{P}^1$ given by homogeneous cubic $L(s^3, s^2t, st^2, t^3)$. [Check - recall φ an isomorphism]. Ie, $\deg V = 3$.

Plane Curves.

If $f, g \in k[x,y]$ irreducible, $U = V(f)$, $W = V(g) \subset \mathbb{A}^2$ and P is a smooth point of U and W, then $\mathcal{O}_{U,P} = \mathcal{O}_{\mathbb{A}^2,P}/(f)$ and $\mathcal{O}_{W,P} = \mathcal{O}_{\mathbb{A}^2,P}/(g)$.
[Recall: $k[U] = k[x,y]/(f)$; similarly for $k[W]$].

If $t \in \mathcal{O}_{U,P}$, local parameter and write $g = ut^n$ in $\mathcal{O}_{U,P}$ (u a unit), then $v_{U,P}(g) = n = \dim \mathcal{O}_{U,P}/(t^n)$ (See Sheet 2, question 1) $= \dim \mathcal{O}_{U,P}/(g) = \dim \mathcal{O}_{\mathbb{A}^2,P}/(f,g)$
$\therefore$ $v_{U,P}(g) = v_{W,P}(f)$.

Suppose now $V \subset \mathbb{P}^2$, defined by irreducible homogeneous polynomial $F(X_0, X_1, X_2)$ of degree $d > 0$, and $L \subset \mathbb{P}^2$ is the line $X_2 = 0$ (wlog $F \neq X_2$).
Above argument shows that $\deg V =$ degree of (X_2) on V = degree of (F) on L = degree of $(F(X_0, X_1, 0))$ on $\mathbb{P}^1 = d = \deg F$.

Definition: Given divisor D on V, define $L(D) = \{f \in k(V)^* \mid (f) + D \geq 0\} \cup \{0\}$
Ie, if $D = \sum n_i P_i$, $0 \neq f \in L(D) \iff v_{P_i}(f) \geq -n_i \ \forall i$ and f regular elsewhere.

Note: If $D_1 - D_2 = (g)$, then multiplication by g defines an isomorphism $L(D_1) \xrightarrow{\cong} L(D_2)$.
ie, $(f) + D_1 \geq 0 \iff (fg) + D_2 \geq 0$.

Define $l(D) = \dim_k L(D)$ [depends only on divisor class].
$l(D) > 0 \iff \exists D' \sim D$ such that $D' \geq 0$.

Lemma 2.1: For $D \geq 0$, $l(D-P) \geq l(D) - 1$.
Proof: Suppose $D = \sum n_i P_i$ $(n_i \geq 0)$. Set $n = n_i$ if $P = P_i$ and $n = 0$ otherwise.
Let t be a local parameter at P. Then $\exists$ linear map $\theta : L(D) \to \mathcal{O}_{V,P}/(t) \cong k$, given by $f \mapsto t^n f = g \mapsto g(P)$. $\operatorname{Ker}\theta = L(D-P)$ and so $\dim L(D)/L(D-P) \leq 1$

Remark: So for V smooth projective curve, noting that $L(D') = 0$ if $\deg D' < 0$, ($\nexists$ any linearly equivalent effective divisor), deduce from lemma 2.1 that $l(D) \leq \deg D + 1$. So, given any divisor D, with $l(D) > 0$, can choose basis $f_0, \dots, f_m$ for $L(D)$ and define rational map (and hence a morphism) $\varphi_D : V \to \mathbb{P}^m$ by $\varphi_D = (f_0 : f_1 : \dots : f_m)$

Note: φ_D depends only on the divisor class of D, ie, that $(gf_0 : \dots : gf_m)$ defines same rational map for any $g \in k(V)^*$.
cf. case of $\mathbb{P}^1$ where φ_D depended only on $\deg D$. [recall $Cl(\mathbb{P}^1) \cong \mathbb{Z}$].

Suppose $V \subset \mathbb{P}^n$ a smooth curve not contained in a hyperplane and D hyperplane section $X_0 = 0$, say. Have independent elements $1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}$ of $L(D)$. If these also span $L(D)$, then $\varphi_D = (1 : \frac{x_1}{x_0} : \dots : \frac{x_n}{x_0}) = (x_0 : x_1 : \dots : x_n)$, the original embedding. In this situation, general non-zero element of $L(D)$ is of the form $h = \frac{L(x_0, \dots, x_n)}{x_0}$, where L is a linear homogeneous form.
Then $\mathbb{P}(L(D)) \longleftrightarrow$ hyperplane sections of V, $h = \frac{L}{x_0} \leftrightarrow (L) = D + (h)$.
Under this correspondance, the subspace $L(D-P)$ corresponds to hyperplanes containing P and $L(D-P-Q)$ to those containing P and Q, (or if $P = Q$, the hyperplanes which intersect V twice at P, ie hyperplanes $L = 0$ such that $v_P(\frac{L}{x_i}) \geq 2$ [where assuming $P \in V \cap \{X_i \neq 0\}$])
When $P \neq Q$, clear that $l(D-P-Q) = l(D) - 2$.

Remark: If $V_0 \subset \mathbb{A}^n$ and $P = (0, \dots, 0) \in V_0$, the space of linear forms defining hyperplanes through P is spanned by $x_1, \dots, x_n$. Moreover, for at least one x_i, must have $x_i \notin m_P^2$ (since $m_P = (x_1, \dots, x_n)$ in $\mathcal{O}_{V,P}$). So $v_P(x_i) = 1$).

Subspace corresponds to hyperplanes having a multiple intersection at P, corresponding to kernel of surjective homomorphism to m_P/m_P^2 where $m_P/m_P^2 \cong \mathcal{O}_{V,P}/m_P \cong k$.
Clear also that $l(D-2P) = l(D) - 2$.
In fact, $l(D-P-Q) = l(D) - 2$ $\forall$ $P, Q \in V$ is enough to ensure that φ_D gives an embedding of V with $D \sim$ hyperplane section, and if we take $D = (X_0)$, then $L(D)$ has basis $1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}$.

Lemma 2.2: If $l(D-P-Q) = l(D) - 2 \;\; \forall P, Q \in V$ (not necessarily distinct) then $\varphi_D : V \to \mathbb{P}^{l(D)-1}$ is an embedding whose image has degree $\deg D$.

If $P \neq Q \in V$ with $P \sim Q$ then $l(P) > 1 \underset{(\text{Lemma } 2.1)}{\Rightarrow} l(D) = 2$, and lemma 2.2 $\Rightarrow \varphi_D : V \to \mathbb{P}^1$ is an embedding and hence an isomorphism.

3. Differentials.

V irreducible smooth projective curve. Define $k(V)$-vector space $\Omega'_{k(V)/k}$ of rational differentials to consist of finite sums $\sum f_i dg_i$ $(f_i, g_i \in k(V))$ subject to the relations:

(i) $da = 0$ for $a \in k$

(ii) $d(f+g) = df + dg$

(iii) $d(fg) = f dg + g df$.

Exercise: for $f \in k(V)$, $g \in k(V)^*$, show that $d\left(\frac{f}{g}\right) = \dfrac{g df - f dg}{g^2}$

For V a curve, $k(V)$ is a finite extension of $k(t)$, for any non-constant t.
Any non-constant $g \in k(V)$ satisfies an irreducible equation over $k(t)$. $F(g) = g^n + a_{n-1}(t) g^{n-1} + \dots + a_0(t) = 0$.
Assuming $\operatorname{char} k = 0$ (eg. $k = \mathbb{C}$), F' is a non-zero polynomial and so coprime to F
$\Rightarrow$ some combination of F and F' is 1 in $k(t)[X]$. $\Rightarrow F'(g) \neq 0$ in $k(V)$.
$0 = dF(g) = F'(g) dg + h dt$, where $h = a'_{n-1}(t) g^{n-1} + \dots + a'_0(t)$, and so dg is a multiple of dt. Ie, $\Omega'_{k(V)/k}$ is 1-dimensional, generated by dt for any non-constant $t \in k(V)$.

Given a non-zero rational differential ω on V and a point $P \in V$, choose local parameter $t \in m_{V,P}$. Writing $\omega = f dt$, $(f \in k(V))$, and defining $v_P(\omega) = v_P(f)$ – we need to check this is independent of choice of local parameter.

Lemma 3.1: (i) The numbers $v_P(dh)$ for $h \in \mathcal{O}_{V,P}$ are bounded below.

(ii) $v_P(dh) \geq 0 \;\; \forall h \in \mathcal{O}_{V,P}$.

(iii) $v_P(dt') = 0$ for any local parameter t' at P.

Thus our definition of $v_P(\omega)$ is independent of the choice of t.

Proof: (i), (ii) exercise

(iii) Write $t' = ut$, u a unit in $\mathcal{O}_{V,P}$. $\therefore dt' = u dt + t du$

(ii) $\Rightarrow du = h dt$ with $v_P(h) \geq 0$. $\therefore dt' = (u + th) dt$

$\therefore v_P(dt') = v_P(u+th) = 0$.

Definition: ω is regular at P if $v_P(\omega) \geq 0$

For $V \subset \mathbb{P}^n$, smooth irreducible projective curve (not contained in a hyperplane, wlog), we can take affine coordinates $x_1, \dots, x_n$ on $\mathbb{A}^n_0$ and observe $\Omega'_{k(V)/k}$ is generated by any one of the dx_i.

Moreover, for any i, lemma 3.1 $\Rightarrow$ $v_P(dx_i) \geqslant 0$ $\forall$ $P \in V_0$. Hence, for w a non-zero rational differential on V, $v_P(w) \geqslant 0$ for all but finitely many $P \in V$.

Lemma 3.2: With notation as above, $v_P(w) = 0$ for all but finitely many $P \in V$

Proof: Enough to show $v_P(dx_1) = 0$ for all but finitely many $P \in V_0$. (Recall: $V \setminus V_0$ is finite)
Since $k(V)/k(x_i)$ finite for each i, $\exists$ irreducible polynomial $f_i(x_1, x_i) \in I(V_0)$, (Recall: V_0 irreducible and $I(V_0)$ prime), of form $a_{i n_i}(x_1)x_i^{n_i} + \dots + a_{i1}(x_1)x_i + a_{i0}(x_1)$ of minimal degree n_i in x_i. Thus, $\frac{\partial f}{\partial x_i} \notin I(V_0)$ (char $k = 0$), and more precisely $\exists$ only finitely many x_1-coordinates for points $P \in V_0$ with $\frac{\partial f_i}{\partial x_i}(P) = 0$ by Lemma 1.2 $\left(\alpha f_i + \beta \frac{\partial f}{\partial x_i} = h(x_1)\right)$.
Each x_1 coordinate a_1 corresponds to only finitely many points $P = (a_0, a_1, \dots, a_n)$ of curve (cf Claim 1 of proposition 1.5). Thus except for finitely many $P \in V_0$, have $\frac{\partial f_i}{\partial x_i}(P) \neq 0$ $\forall i > 1$.

Claim: $v_P(dx_1) = 0$ for all such points.

Proof: Observe $\frac{\partial f_i}{\partial x_1} dx_1 + \frac{\partial f_i}{\partial x_i} dx_i = 0$ in $\Omega^1_{k(V)/k}$ and so if $v_P(dx_1) > 0$, deduce $v_P(dx_i) > 0$ $\forall i$. This however contradicts the fact that P is smooth (wlog $P = (0, \dots, 0)$, then $v_P(dx_j) > 0$ $\forall j \Rightarrow x_j \in m^2_{V,P}$ $\forall j$ $\divideontimes$, since $m_{V,P} = (x_1, \dots, x_n)$ and local parameter $t \in m_{V,P}/m^2_{V,P}$)

The lemma is true for dx_1, and since $\Omega^1_{k(V)/k}$ is 1-dimensional over $k(V)$, the lemma is true for any rational differential.

Definition: Define the divisor (w) of $w \neq 0$ by $(w) = \sum_{P \in V} v_P(w) P$, usually denoted by K_V, the canonical divisor.

Any other rational differential $w' \neq 0$ is of the form $w' = hw$ for some $h \in k(V)^*$ and so $(w') = (h) + (w)$. I.e, we have a uniquely defined divisor class on V, also denoted K_V, and called the canonical class.
For V, smooth irreducible projective curve, we therefore have an invariant $\deg(K_V)$.
This is invariant under isomorphisms.
An isomorphism $\varphi: V \to W$ induces not only an isomorphism of function fields, $\varphi^*: k(W) \to k(V)$, but also an isomorphism of local rings, $\varphi^*: \mathcal{O}_{W,\varphi(P)} \to \mathcal{O}_{V,P}$.
The obvious induced isomorphism $\varphi^*: \Omega^1_{k(W)/k} \to \Omega^1_{k(V)/k}$ has the property that for $w = \sum f_i\, dg_i$ on W, $\varphi^*(w) = \sum (\varphi^* f_i)\, d(\varphi^* g_i)$ on V, and furthermore that $v_P(\varphi^*(w)) = v_{\varphi(P)}(w)$ $\forall$ $P \in V$. Thus $\deg K_V = \deg(\varphi^* w) = \deg(w) = \deg K_W$.
For V as above can consider vector space over k of everywhere regular rational differentials, ie, $(w) \geqslant 0$.

If w_0 fixed, non-zero rational differential with $K_V = (w_0)$, then an arbitrary non-zero rational differential $w = hw_0$ ($h \in k(V)^*$) is regular everywhere iff $h(w_0) = (h) + K_V \geqslant 0$.
Ie, $h \in L(K_V)$. The space of global regular differentials on V is isomorphic to $L(K_V)$ by $h \in L(K_V) \leftrightarrow hw_0$. This space has finite dimension $l(K_V)$.
By definition, this is the genus of V (clearly invariant under isomorphism), denoted $g(V)$.
We will see later that $\deg K_V = 2g(V) - 2$.

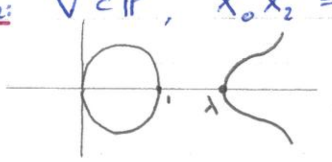
Example: $V = \mathbb{P}^1$, affine piece $\mathbb{A}^1$, with affine coordinate $x = \frac{X_1}{X_0}$. Observe $x - x(P)$ is a local parameter at P $(\forall P \in \mathbb{A}^1)$ and so $v_P(dx) = 0$ $\forall P \in \mathbb{A}^1$.
Local parameter at ∞ is $y = \frac{1}{x}$. But $dx = d\left(\frac{1}{y}\right) = -\frac{1}{y^2}dy \Rightarrow v_\infty(dx) = -2$
Ie, $K_{\mathbb{P}^1} = -2P_\infty$. In particular, $l(K_{\mathbb{P}^1}) = 0$, ie $g(\mathbb{P}^1) = 0$.

Definition: An irreducible curve V is said to be rational if its function field $k(V) \cong k(\mathbb{P}^1) \cong k(t)$

Example: the cuspoidal cubic, $ZY^2 = X^3$: and the nodal cubic, $ZY^2 = X^2(X+Z)$: are rational curves (exercise - look at affine pieces).

If V is a smooth projective curve with $k(V) \cong k(t)$, let $h \in k(V)$ be the rational function corresponding to t and $\varphi = (1:h) : V \to \mathbb{P}^1$, non-constant morphism.
Then $\varphi^*(x) = h$ and so $\varphi^* : k(\mathbb{P}^1) \cong k(x) \xrightarrow{\cong} k(V)$. Question 3 on sheet 2 (with φ non-constant rather than surjective) $\Rightarrow \varphi : V \xrightarrow{\cong} \mathbb{P}^1$. Hence $g(V) = 0$.

Example: $V \subset \mathbb{P}^2$, $X_0 X_2^2 = X_1(X_1 - X_0)(X_1 - \lambda X_0)$, $y^2 = x(x-1)(x-\lambda)$.



$g(V) = 1$ - see printed sheet (page 7b)
$\frac{dx}{y}$ is everywhere regular on V with $v_P\left(\frac{dx}{y}\right) = 0$ $\forall P$.
Hence $\left(\frac{dx}{y}\right) = 0 = K_V$ and so $g(V) = l(0) = 1$ (by lemma 2.1)

Definition: A curve of genus 1 is called elliptic.

Note that in the above example, $\exists$ degree-2 morphism $\pi : V \to \mathbb{P}^1$ $(\pi = (X_0 : X_1))$.

For any smooth projective curve V, say that V is hyperelliptic if $\exists$ degree 2 morphism $\pi : V \to \mathbb{P}^1$.
Take $V \subset \mathbb{P}^2$, a smooth plane projective curve, defined by irreducible homogeneous polynomial $F(X_0, X_1, X_2)$ of degree d.

Proposition 3.3: With V as above, $K_V \sim (d-3)H_V$, where H_V is divisor on V cut out by $X_0 = 0$, ie $H_V = (X_0)$. Then, $\deg K_V = (d-3)d$.

Proof: Let U be an affine piece of V given by $X_0 \neq 0$. On U have affine coordinates $y_1 = \frac{X_1}{X_0}$, $y_2 = \frac{X_2}{X_0}$, and $U \in \mathbb{A}^2$ defined by $f \in k[y_1, y_2]$, polynomial of degree d, where $f(y_1, y_2) = X_0^{-d} F(X_0, X_1, X_2)$. In $\Omega^1_{k(V)/k}$ we have $0 = df = \frac{\partial f}{\partial y_1}dy_1 + \frac{\partial f}{\partial y_2}dy_2$. ⊛
Let $U_1 = \{P \in V$ such that $\frac{\partial f}{\partial y_1}(P) \neq 0\}$, $U_2 = \{P \in V$ such that $\frac{\partial f}{\partial y_2}(P) \neq 0\}$.
Since V smooth, $U = U_1 \cup U_2$.
For $P \in U$, $v_P(dy_1) = v_P\left(\frac{\partial f}{\partial y_1}dy_1\right) = v_P\left(-\frac{\partial f}{\partial y_2}dy_2\right) \geq v_P(dy_2)$. Since one of $y_1 - y_1(P)$ and $y_2 - y_2(P)$ has to be a local parameter at P, one of the $v_P(dy_i)$ is 0.
But for $P \in U_1$, $v_P(dy_2) > 0 \Rightarrow v_P(dy_1) > 0$ ⚹ $\therefore$ $v_P(dy_2) = 0$ $\forall P \in U_1$.
Similarly, $v_P(dy_1) = 0$ $\forall P \in U_2$.
If we consider rational differential $\omega = -\frac{dy_2}{\partial f/\partial y_1} = \frac{dy_1}{\partial f/\partial y_2}$ (from ⊛), $\in \Omega^1_{k(V)/k}$.
This has $v_P(\omega) = 0$ for $P \in U_1$ and $P \in U_2$, ie $\forall P \in U$. (cont.)

We therefore need to consider form of ω at ∞; wlog can assume $(0:0:1) \notin V$ and denote by $W \subset \mathbb{A}^2$ the affine piece given by $X_1 \neq 0$ (ie. $V = U \cup W$), and have coordinates $(z_1, z_2) = \left(\frac{1}{y_1}, \frac{y_2}{y_1}\right)$. $z_1 = \frac{X_0}{X_1}$, $z_2 = \frac{X_2}{X_1}$, and W defined by an equation: $g(z_1, z_2) = z_1^d f\left(\frac{1}{z_1}, \frac{z_2}{z_1}\right)$.
Since $y_1 = \frac{1}{z_1}$, $dy_1 = -\frac{dz_1}{z_1^2}$. Also, $\frac{\partial g}{\partial z_2} = z_1^{d-1} \cdot \frac{\partial f}{\partial y_2}$.
Defining w_1, w_2 in an analogous way to u_1, u_2, we have $\omega = -z_1^{d-3} \cdot \frac{dz_1}{\partial g / \partial z_2}$.
Since $\frac{\partial g}{\partial z_1} dz_1 + \frac{\partial g}{\partial z_2} dz_2 = 0$ in $\Omega^1_{k(V)/k}$, have $\omega = z_1^{d-3} \cdot \frac{dz_2}{\partial g / \partial z_1}$.
But $\omega_0 = \frac{dz_2}{\partial g / \partial z_1} = -\frac{dz_1}{\partial g / \partial z_2}$ has the property that $v_P(\omega_0) = 0 \ \forall P \in W$.
Since $\omega = z_1^{d-3} \cdot \omega_0$, have $(\omega) = (z_1^{d-3})$ on the affine piece W.
Recalling $(0:0:1) \notin V$, deduce $(\omega) = (X_0^{d-3})$ on V. Ie $(\omega) = (d-3) H_V$, where $H_V = (X_0)$, hyperplane section.

Remark: The general identity: $\deg K_V = 2g - 2$ (which we will see in section 4) then implies that $g(V) = \frac{1}{2}(d-1)(d-2)$. (so only triangular numbers ~~be~~ occur here).

Corollary: Any smooth cubic has $K_V \sim 0$ and hence $g = l(0) = 1$ (by lemma 2.1) and hence elliptic. A smooth plane curve is rational iff degree $= 1, 2$ (since $d > 2 \Rightarrow K_V \geqslant 0$)

4. Riemann-Roch and consequences.

Riemann-Roch Theorem: Given smooth projective curve V of genus g and divisor D on V, $l(D) = 1 - g + \deg D + l(K_V - D)$.

Another way of writing this: $l(D) - l(K_V - D) = l(0) - l(K_V - 0) + \deg D$.
cf: lemma 2.1: $l(D) \leq 1 + \deg D$ for $D \geqslant 0$, so it is $l(D) - l(K_V - D)$, which behaves well.

Corollary: For V a smooth projective of genus g, $\deg K_V = 2g - 2$.
Proof: Put $D = K_V$ in Riemann-Roch: $g = 1 - g + \deg K_V + l(0)$, and $l(0) = 1$

Proposition 4.2: A smooth projective curve V has genus $0 \iff V \cong \mathbb{P}^1$.
Proof: $K=$) from section 3.
($\Rightarrow$) Given $P \in V$, observe: $l(P) = 1 + 1 + \underbrace{l(K_V - P)}_{\deg = -3} = 2$. Now apply question 4 from sheet 2.

Pullbacks of differentials

Recall: given non-constant morphism $\varphi: V \to W$ between smooth projective curves, have $\varphi^*: k(W) \hookrightarrow k(V)$, and this induces a homomorphism of $k(W)$-vector spaces $\varphi^*: \Omega^1_{k(W)/k} \to \Omega^1_{k(V)/k}$. If $\omega = \sum f_i \, dg_i$ then $\varphi^*(\omega) = \sum \varphi^*(f_i) \, d(\varphi^* g_i)$.
In characteristic 0, this homomorphism is injective. (Recall that for any non-constant $g \in k(V)$ in characteristic 0, dg spans $\Omega^1_{k(V)/k}$, and so in particular for non-constant $h \in k(W)$, $d(\varphi^* h) \neq 0$ in $\Omega^1_{k(V)/k}$.

Remark: This fails for characteristic p, eg morphism $\varphi: \mathbb{P}^1 \to \mathbb{P}^1$, $(x_0 : x_1) \mapsto (x_0^p : x_1^p)$ induces map on function fields, $k(x) \hookrightarrow k(x)$, $x \mapsto x^p$.
The corresponding map φ^* on differentials is then zero: $\varphi^*(dx) = dx^p = 0$.

If $\varphi(P) = Q$ then $\varphi^*: \mathcal{O}_{W,Q} \hookrightarrow \mathcal{O}_{V,P}$. If w regular at Q then $w = f\,dt$ for some local parameter t at Q and some $f \in \mathcal{O}_{V,Q}$. Then $\varphi^*(w) = \varphi^*(f)\, d\varphi^*(t)$ regular at P (by lemma 3.1). So if Ω'_W (respectively Ω'_V) denotes the k-vector space of everywhere regular differentials, have induced injective linear map $\varphi^*: \Omega'_W \hookrightarrow \Omega'_V$.
In particular, $g(W) \leq g(V)$ [cf. Riemann-Hurwitz later].
Thus if $\varphi: V \to W$, non-constant morphism of smooth projective curves, then V rational $\Rightarrow g(V) = 0 \Rightarrow g(W) = 0 \Rightarrow W$ rational. (Geometric form of Luroth's Theorem).

Elliptic Curves.

Let V be a smooth projective curve of genus 1 and $P_0 \in V$, some fixed point.
For D a divisor of degree 0 on V, Riemann-Roch $\Rightarrow l(D+P_0) = 1 - 1 + 1 + \underbrace{l(K_V - D - P_0)}_{deg = -1} = 1$.
I.e, $\exists$ unique $P \in V$ such that $D + P_0 \sim P$, ie. such that $D \sim P - P_0$.
Denote by $Cl^0(V)$ the divisor classes of degree 0 on V.
We have map: $V \to Cl^0(V)$ given by: $P \mapsto$ class $(P - P_0)$, which is a bijection between points of V and the elements of $Cl^0(V)$.
The abelian group structure on $Cl^0(V)$ therefore induces an abelian group structure on the points of V, with identity P_0.

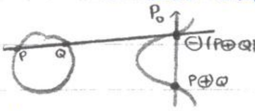
Recall now, embedding criterion: if D divisor on smooth projective curve V such that $l(D - P - Q) = l(D) - 2 \ \forall\ P, Q \in V$ then $\varphi_D: V \hookrightarrow \mathbb{P}^{l(D)-1}$ is an embedding whose image $\varphi(V)$ has degree $\deg D$.

For V elliptic and $P_0 \in V$, take $D = 3P_0$. Then $l(D - P - Q) = 0 + 1 + \underbrace{l(K_V - D + P + Q)}_{deg = -1} = 1$, and $l(D) = 0 + 3 + \underbrace{l(K_V - D)}_{deg = -3} = 3$
Embedding criterion satisfied, so $\varphi_D: V \hookrightarrow \mathbb{P}^2$ as a smooth plane cubic.
If we pick a basis $\{1, f_1, f_2\}$ for $L(3P_0)$, and set $\varphi_{3P_0} = (1 : f_1 : f_2)$, it is clear that P_0 is the only point of V mapping to ∞, (ie $X_0 = 0$); for $P \neq P_0$, f_1, f_2 are regular at $P \Rightarrow \varphi_{3P_0}(P) = (1 : f_1(P) : f_2(P))$. That is, the line $\{X_0 = 0\}$ intersects $\varphi(V)$ only at $\varphi(P_0)$ and then the divisor cut out by this line, $(X_0) = 3\varphi(P_0)$ on V.
Ie, by definition, $\varphi(P_0)$ is an inflexion point of embedded curve $\varphi(V)$.
Observe that three points $P, Q, R \in V$ add to zero in the induced group law $\Leftrightarrow$ $(P - P_0) + (Q - P_0) + (R - P_0) = 0$ in $Cl^0(V)$ $\Leftrightarrow$ $P + Q + R \sim 3P_0$ as divisors on V.

Claim: this happens iff divisor $\varphi(P) + \varphi(Q) + \varphi(R)$ is cut out by a line on $\varphi(V)$.
Ie, points $\varphi(P), \varphi(Q), \varphi(R)$ are collinear on $\varphi(V)$, where $\varphi = \varphi_{3P_0} = (1 : f_1 : f_2)$ is the above embedding.

$\varphi_{3P_0} : V \hookrightarrow \mathbb{P}^2$. If we take basis $1, f_1, f_2$ for $L(3P_0)$ then $\varphi = (1 : f_1 : f_2)$ has the property that P_0 is the only point mapping to line at ∞, $X_0 = 0$. Ie, $(X_0) = 3\varphi(P_0)$, ie $\varphi(P_0)$ is an inflection point. If we identify V with its image under $\varphi = \varphi_{3P_0}$ and so $(X_0) = 3P_0$, and any other hyperplane section is of the form: $(L) = (\frac{L}{X_0}) + 3P_0$ for some linear form L. Since $l(3P_0) = 3$, observe that $L(3P_0) = \{ {}^{L}/_{X_0} \in k(V) \}$, ie the hyperplane sections of V are precisely the effective divisors linearly equivalent to $3P_0$.

Thus P, Q, R add to 0 in the group law on $V \iff P + Q + R$ is cut out by a line, ie the three points are collinear. Have picture:



(P_0 inflexion point).

Let $\oplus, \ominus$ denote addition and subtraction in induced group law.

Legendre Normal Form for Elliptic Curve V.

Riemann-Roch $\Rightarrow$ $l(2P_0), l(3P_0) = 3$. Choose basis $\{1, x\}$ for $L(2P_0)$ and extend to basis $\{1, x, y\}$ for $L(3P_0)$, Ie, $v_{P_0}(x) = -2$, $v_{P_0}(y) = -3$.

We now take $\varphi = (1 : x : y)$. Riemann-Roch $\Rightarrow$ $l(6P_0) = 6$. Note that the seven functions $1, x, y, x^2, xy, x^3, y^2$ are all in $L(6P_0)$, so are linearly dependent. Since x^3, y^2 are the only ones with 6-fold pole at P_0, both occur with non-zero coefficients in this relation. Replacing x, y by suitable multiples, may assume that the linear relation is of the form:

$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$, for suitable $a_i \in k$, ie the affine ~~equation~~ image of V (ie, of $V \setminus \{P_0\}$) satisfies an equation of the above form.

Now make obvious changes of coordinates (ie corresponding to different choices of x and y) to bring equation into the form $y^2 = f(x)$. [f a cubic with distinct roots as V smooth], and then into the form $y^2 = x(x-1)(x-\lambda)$, $(\lambda \neq 0, 1)$, ie, projective image of V has equation $X_0 X_2^2 = X_1(X_1 - X_0)(X_1 - \lambda X_0)$ $(\lambda \neq 0, 1)$. This is the Legendre Normal Form for V.

Recall: If $\Lambda \subset \mathbb{C}$ is a lattice with corresponding Weierstrass p-function $\wp$, then $(1 : \wp : \wp')$ embeds $\mathbb{C}/\Lambda$ in $\mathbb{P}^2(\mathbb{C})$. We have relation between $\wp$ and $\wp'$ of the form $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ and so embedded torus $\mathbb{C}/\Lambda$ has equation $X_0 X_2^2 = 4X_1^3 - g_2 X_1 X_0^2 - g_3 X_0^3$. (Weierstrass Normal Form).

In particular, the Legendre Normal Form exhibits V as a double cover of $\mathbb{P}^1$ branched over four points $0, 1, \lambda, \infty$, where double cover map is just $\pi = (1 : x)$.

From the geometrical description of group law on V, it follows that the map (for fixed $Q \in V$) $\varphi : V \to V$ given by $\varphi(P) = P \oplus Q$ is a rational map and hence a morphism. This is clear, since for fixed $R \in V$ the map $\Psi : V \to V$ given by $P \mapsto$ third point of intersection of RP with V is a rational map.

If V is given by $F(X_0, X_1, X_2) = 0$ and $R = (a : b : c)$ and $P = (x_0 : x_1 : x_2)$, the line RP is given parametrically by $(\lambda a + \mu x_0 : \lambda b + \mu x_1 : \lambda c + \mu x_2)$, $(\lambda : \mu) \in \mathbb{P}^1$, and $F(\lambda a + \mu x_0, \lambda b + \mu x_1, \lambda c + \mu x_2)$ * is divisible be $\lambda\mu$, and hence can solve (from remaining linear factor in λ, μ) for $(\lambda : \mu)$ in terms of polynomial functions in $(x_0 : x_1 : x_2)$.

Hence Ψ a rational map, hence a morphism, with $\Psi(V \setminus \{R\}) = V \setminus \{R'\}$

Finiteness $\Rightarrow$ Ψ surjective and hence $\Psi(R) = R'$ as required.

Since φ is the composite of two such maps, φ is a morphism. φ clearly has an inverse and so an isomorphism. Hence, group of automorphisms of V, $Aut(V)$ is transitive.

Remark: for $g>1$, $Aut(V)$ is finite. In fact $|Aut(V)| \leq 84(g-1)$.

Example: $X_0^3 X_1 + X_1^3 X_2 + X_2^3 X_0 = 0$, (The Klein quartic), has $g=3$ and $|Aut(V)| = 168$.

Ramification and Riemann-Hurwitz.

Let $\varphi: V \to W$ be a non-constant morphism of degree n of smooth projective curves. Already seen $g(W) \leq g(V)$.

Definition: For $P \in V$, define the ramification index e_P of φ at P as follows:
Let $Q = \varphi(P) \in W$ and t local parameter at Q. Then $e_P := v_P(\varphi^*(t))$.
(clearly independent of choice of t).

Finiteness theorem: $n = \sum_{P \in \varphi^{-1}(Q)} e_P$.
If $e_P > 1$, say that φ is ramified at P and Q is a branch point. If $e_P = 1$, φ is unramified at P. Observe that for any local parameter t at Q, then $\varphi^*(t) = us^{e_P}$ for s local parameter at P and u a unit in $\mathcal{O}_{V,P}$.
$\therefore \varphi^*(dt) = d(\varphi^*(t)) = d(us^{e_P}) = e_P u s^{e_P - 1} ds + s^{e_P} du. \Rightarrow v_P(\varphi^*(dt)) = e_P - 1$ by Lemma 3.1 (ii) and char $= 0$. In particular, φ unramified at $P \Leftrightarrow v_P(\varphi^*(dt)) = 0$.

Claim: φ has only finitely many ramification points.
Proof: For $t \in k(W) \setminus k$, know that if t regular at $Q \in W$, then $t - t(Q)$ is a local parameter at $Q \Leftrightarrow v_Q(dt) = 0$. But t is non-regular at only finitely many points of W, and by Lemma 3.2, $v_Q(dt) = 0$ for all but finitely many $Q \in W$, ie, $t - t(Q)$ is a local parameter at Q for all but finitely many $Q \in W$.
Lemma 3.2 $\Rightarrow v_P(\varphi^* dt) = 0$ for all but finitely many $P \in V$.
Hence we deduce that there are only finitely many ramification points for ~~Q~~. φ.

Theorem 4.3 (Riemann-Hurwitz): Let $\varphi: V \to W$ be a morphism of degree n between smooth projective curves. Then $2g(V) - 2 = n(2g(W) - 2) + \sum_{P \in V} (e_P - 1)$.

Example: If $\pi: V \to \mathbb{P}^1$ double cover (ie, V hyperelliptic) then π is branched over $2g(V)+2$ points. (Since π double cover, $e_P = 1, 2$ and $2g(V) - 2 = -4 + \#$ ramification points.

Proof: Since $\exists$ only finitely many branch points $Q \in W$, we can choose $w \in \Omega^1_{k(W)/k}$ such that $v_Q(w) = 0$ for branch points Q.
$\ulcorner$ Take an affine piece W_0 of W which contains all branch points. For each branch point $Q_i \in W$, we can find an element $h_i \in k[W_0]$ which is a local parameter at Q_i but non-vanishing at the other branch points. Eg, show that $\exists$ a suitable affine hyperplane through Q_i but not containing any of the other branch

points whose equation is a local parameter at Q_i, ie look at the linear map: $\{$linear forms vanishing at $Q_i\} \to {}^{m_{Q_i}}/_{m^2_{Q_i}} \cong k$, and take linear form not in kernel and not vanishing at the branch points$]$

$\therefore$ For w_0 any non-zero rational differential, can achieve W with required property by multiplying by powers of the h_i. Considering non-zero differential $\varphi^* w$ on V, $2g(V) - 2 = \deg(\varphi^* w)$ (by Riemann-Roch) $= n \deg w + \{$contribution from ramification points$\}$. [Since, if Q is not a branch point of φ then $v_P(\varphi^* w) = v_Q(w)$ for all n points P (distinct) with $\varphi(P) = Q$, since $v_P(\varphi^* dt) = 0$ for any local parameter t at Q].

Hence poles and zeroes of w give rise to corresponding poles and zeroes of $\varphi^* w$ at the corresponding n distinct points.

If P is a ramification point with $\varphi(P) = Q$ then $v_P(\varphi^* W) = v_P(\varphi^* dt) = e_P - 1$, from previous calculation (t local parameter at Q).

Thus, $2g(V) - 2 = n(2g(W) - 2) + \sum_{P \in V} (e_P - 1)$.

Corollary*: For V a smooth complex projective curve, the geometric genus $g(V)$ we have defined = topological genus. Ie, topologically, V is a sphere with g handles.

Double Cover.

Let $\pi : V \to \mathbb{P}^1$ be a hyperelliptic curve of genus $g > 0$, π of degree 2, and let $P \in V$ be a ramification point and $Q = \pi(P)$. Take $t \in L(Q) \setminus k$, $t \in k(\mathbb{P}^1)$. If $Q = (q_0 : q_1)$, then $t = \frac{c(v_1 x_0 - v_0 x_1)}{q_1 x_0 - q_0 x_1}$. Thus V_t is a local parameter at Q, and $v_P(\pi^* t) = 2$, (ie $v_P(\pi^*(v.t)) = 2$), and $\pi^*(t) = L(2P) \subset k(V)$.

But $L(2P) \leq 2$ (by sheet 2, question 5), and so $\{1, \pi^* t\}$ is a basis of $L(2P)$.

Now, $\varphi_{2P} = (1 : \pi^* t) = (1 : \psi_t) \circ \pi$, where $\psi_t = (1:t) : \mathbb{P}^1 \to \mathbb{P}^1$ and is a linear automorphism of $\mathbb{P}^1$, ie, is an element of $PGL(2, \mathbb{C})$, ie, up to a (linear) automorphism of $\mathbb{P}^1$, we have $\varphi_{2P} = \pi$.

Proposition 4.4: Given an elliptic curve V and two double covers $\pi_i : V \to \mathbb{P}^1$, $i = 1, 2$, $\exists$ automorphism $\sigma \in Aut(V)$ and linear automorphism $\tau \in Aut(\mathbb{P}^1)$ such that $\pi_2 \circ \sigma = \tau \circ \pi_1$.

$$\begin{tikzcd} V \arrow[r, "\sigma"'] \arrow[d, "\pi_1"'] & V \arrow[d, "\pi_2"] \\ \mathbb{P}^1 \arrow[r, "\tau"] & \mathbb{P}^1 \end{tikzcd}.$$

Proof: Let P_i be ramification points for $\pi_i : V \to \mathbb{P}^1$ $(i = 1, 2)$. Since $Aut(V)$ transitive, can choose $\sigma \in Aut(V)$ such that $\sigma(P_1) = P_2$. Then $\pi_2 \circ \sigma$ is a double cover and P_1 is a ramification point. $\therefore \exists\, \tau_1, \tau_2$ linear automorphisms of $\mathbb{P}^1$ such that $\tau_1 \pi_1 = \varphi_{2P_1} = \tau_2 \pi_2 \sigma$.

Ie, $\tau \pi_1 = \pi_2 \sigma$ for some linear automorphism of $\mathbb{P}^1$ ($\tau = \tau_2^{-1} \tau_1$).

Given an elliptic curve V and $P_0 \in V$, recall $\exists$ double cover $\pi = \varphi_{2P_0} : V \to \mathbb{P}^1$ branched at $0, 1, \lambda, \infty$ with $\pi(P_0) = \infty$. (cf. Legendre Normal Form).

Lemma 4.5: Let S_3 act on $\mathbb{C} \setminus \{0,1\}$ as follows: given $\lambda \in \mathbb{C} \setminus \{0,1\}$, permute the numbers $0, 1, \lambda$ according to $\alpha \in S_3$, and then apply the affine linear transformation sending first to 0, second to 1, and then define $\alpha(\lambda)$ to be the image of the third. The orbit of λ under this action is: $\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda}$

Definition: $j(\lambda) = \frac{2^8(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$.

Check that this is invariant under the above action of S_3. Sufficient to prove $j(\lambda) = j(\frac{1}{\lambda}) = j(1-\lambda)$. Suppose $j = j(\lambda) \in \mathbb{C}$, $j \neq 0, 1$. Consider $2^8(x^2-x+1)^3 - jx^2(x-1)^2 = 0$. Satisfied by λ and its images under the action of S_3. Galois Theory (or direct calculation) $\Rightarrow$ polynomial splits as $2^8(x-\lambda)(x-\frac{1}{\lambda})(x-(1-\lambda))(x-\frac{1}{1-\lambda})(x-\frac{\lambda}{\lambda-1})(x-\frac{\lambda-1}{\lambda})$, ie the solutions of the above equation are all related via above action of S_3. Thus: $j(\lambda') = j(\lambda) \Leftrightarrow \lambda'$ and λ differ by an element of S_3.

V an elliptic curve:

Theorem 4.6: (a) with λ obtained from Legendre normal form V, $j(\lambda)$ depends only on V (ie not on any choices made).

(b) two elliptic curves are isomorphic, $V_1 \simeq V_2 \Leftrightarrow j(V_1) = j(V_2)$.

(c) every element of $\mathbb{C}$ is the j-invariant of some elliptic curve.

Remark: Hence the isomorphism classes of elliptic curves are parametrised by $\mathbb{C}$ (via the j-invariant).

Proof: (a) Suppose we have two different choices of base point, P_1 and $P_2 \in V_1$. Then we obtain double covers $\pi_i : V \to \mathbb{P}^1$ $(i=1,2)$ with $\pi_i(P_i) = \infty$. As in proposition 4.4, choose $\sigma \in \mathrm{Aut}(V)$ such that $\sigma(P_1) = P_2$ and then $\exists$ a linear automorphism τ of $\mathbb{P}^1$ such that $\pi_2\sigma = \tau\pi_1$ and $\tau(\infty) = \infty$. τ sends the other branch points $\{0, 1, \lambda_1\}$ of π_1 to $\{0, 1, \lambda_2\}$ of π_2, in some order.
$$\begin{array}{ccc} V & \xrightarrow{\sigma} & V \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{P}^1 & \xrightarrow{\tau} & \mathbb{P}^1 \end{array}$$

As τ is an affine linear transformation $(\tau\infty = \infty)$ we deduce that λ is defined uniquely up to the action of S_3 defined in lemma 4.5. Thus $j = j(\lambda)$ depends only on V.

(b) Given V_1, V_2 can take Legendre Normal Forms, $X_0X_2^2 = X_1(X_1-X_0)(X_1-\lambda_iX_0)$, $i=1,2$. Recall that $j(\lambda_1) = j(\lambda_2) \Leftrightarrow \lambda_1, \lambda_2$ related by an element of S_3, ie, iff $\exists$ a linear affine change of variables in the coordinates $x = \frac{X_1}{X_0}$ which makes $\lambda_1 = \lambda_2$. Hence, if $j(\lambda_1) = j(\lambda_2)$, can take Legendre Normal Forms for V_1 and V_2 with $\lambda_1 = \lambda_2$. $\therefore V_1$ isomorphic to V_2.

(c) Given any $j \in \mathbb{C}$, can solve the equation $2^8(x^2-x+1)^3 - jx^2(x-1)^2 = 0$, for λ where $\lambda \neq 0, 1$. $\therefore X_0X_2^2 = X_1(X_1-X_0)(X_1-\lambda X_0)$ is an elliptic curve with the required j-invariant.

Classification of curves of small genus

Recall the **Embedding Criterion:** if $l(D-P-Q) = l(D)-2$ $\forall P, Q \in V$ then $\varphi_D : V \to \mathbb{P}^{l(D)-1}$ is an embedding whose image has degree $\deg D$.

Corollary: If D is a divisor on a smooth projective curve of genus g then $d = \deg D \geq 2g+1 \Rightarrow \varphi_D : V \to \mathbb{P}^{d-g}$ is an embedding.

Proof: Riemann-Roch $\Rightarrow l(D) = 1-g+d+\overbrace{l(K_V-D)}^{=0,\ \deg\leq -3} = d-g+1$. $l(D-P-Q) = 1-g+d-2+l(\overbrace{K_V-D+P+Q}^{=0,\ \deg=-1}) = d-g+1$
Hence embedding criterion satisfied.

Proposition 4.7: If V is a smooth projective curve of genus $g \geq 2$, then $\varphi_{K_V}: V \to \mathbb{P}^{g-1}$ is an embedding unless V is hyperelliptic.

Remarks: If V is hyperelliptic of genus $g \geq 2$, sheet 3 question 6 shows that φ_{K_V} is a degree 2 morphism onto a smooth rational curve in $\mathbb{P}^{g-1}$ (the twisted $(g-1)$-ic), and that, unlike the case $g=1$, the double cover map is canonically defined and essentially unique.

Proof: Suppose φ_{K_V} not an embedding. Then lemma 2.2 $\Rightarrow \exists\, P, Q \in V$ such that $l(K_V - (P+Q)) > l(K_V - 2) = g-2$. But $l(P+Q) = 1 - g + 2 + l(K_V - (P+Q))$, by Riemann-Roch, and this is >1. Ie, $\exists f \in L(P+Q)$ non-constant. Let $\varphi = (1:f): V \to \mathbb{P}^1$, non-constant, and P, Q are the only points of V which map to $(0:1) \in \mathbb{P}^1$.
Since $V \not\cong \mathbb{P}^1$, f has a pole at both P and Q (if $P=Q$, then $v_P(f) = -2$), cf sheet 2, question 4. But $t = \frac{x_0}{x_1}$ is a local parameter at $(0:1) \in \mathbb{P}^1$ with $\varphi^*(t) = 1/f$.
Thus, finiteness theorem $\Rightarrow \deg \varphi = 2$. (ie, if $P \neq Q$, $\deg \varphi = 1+1$, if $P = Q$, $\deg \varphi = v_P(\varphi^*(t)) = 2$)
Thus $\varphi: V \to \mathbb{P}^1$ is a double cover and V is hyperelliptic.

Curves of genus 2.

If V a smooth projective curve of genus 2, then $\varphi_{K_V}: V \to \mathbb{P}^1$, so cannot be an embedding. Proposition 4.7 $\Rightarrow V$ hyperelliptic and φ_{K_V} has degree $2g-2 = 2$, and so is a double cover. Riemann-Hurwitz $\Rightarrow \varphi_{K_V}$ branched over 6 points.

Curves of genus 3.

Proposition 4.8: If V is a non-hyperelliptic curve of genus 3 then $\varphi_{K_V}: V \hookrightarrow \mathbb{P}^2$ embeds V as a smooth plane quartic.

Proof: $\varphi_{K_V}: V \to \mathbb{P}^2$ obvious, and it is an embedding by Proposition 4.7, whose image is a smooth curve of degree $\deg K_V = 4$.

Remarks: (i) Already seen that any smooth quartic has genus 3.
(ii) Note that sheet 3 question 10 $\Rightarrow$ a smooth plane quartic is non-hyperelliptic.

Classification Table:

$g=0$	$V \cong \mathbb{P}^1$
$g=1$	V plane cubic, classified by the j-invariant.
$g=2$	V hyperelliptic, double cover of $\mathbb{P}^1$, branched over 6 points.
$g=3$	either V hyperelliptic, or $V \cong$ smooth plane quartic.
$g=4$	either V hyperelliptic, or $V \cong Q \cap W \subset \mathbb{P}^3$, with Q unique irreducible quadric surface and W an irreducible cubic surface. $V \subset \mathbb{P}^3$ is a curve of degree 6.

Fact: For $g > 1$, smooth projective curves of genus g depend on $3g-3$ continuous parameters.

Example: $g=3$, hyperelliptic curves depend on $8-3=5$ parameters. (Branched over 8 points, can make 3 of them $0, 1, \infty$).
{smooth plane quartics}$/GL(3,\mathbb{C})$ depend on $\binom{6}{4}-9 = 15-9 = 6$ parameters, $=3g-3$