1. Construct a 3-regular graph on 8 vertices. Is there a 3-regular graph on 9 vertices?

2. Prove that every connected graph has a vertex that is not a cutvertex.

3. Let G be a graph on n vertices, $G \neq K_n$. Show that G is a tree if and only if the addition of any edge to G produces exactly 1 new cycle.

4. Let $n \geq 2$, and let $d_1 \leq d_2 \ldots \leq d_n$ be a sequence of integers. Show that there is a tree with degree sequence $d_1, \ldots, d_n$ if and only if $d_1 \geq 1$ and $\sum d_i = 2n - 2$.

5. Let $T_1, \ldots, T_k$ be subtrees of a tree T, any two of which have at least one vertex in common. Prove that there is a vertex in all the T_i.

6. Let G be a graph, with degree sequence $d_1 \leq \ldots \leq d_n = \Delta$, such that $d_i \geq i$ for all $i \leq n - \Delta - 1$. Prove that G is connected.

7. The *clique number* of a graph G is the maximum order of a complete subgraph of G. Show that the possible clique numbers for a regular graph on n vertices are $1, 2, \ldots, \lfloor n/2 \rfloor$ and n.

8. Let G be a graph on vertex set V. Show that there is a partition $V_1 \cup V_2$ of V such that in each of $G[V_1]$ and $G[V_2]$ all vertices are of even degree.

9. Let G be a bipartite graph with vertex classes X, Y. Show that if G has a matching from X to Y then there exists $x \in X$ such that every edge incident with x extends to a matching from X to Y.

10. Let G be a connected bipartite graph with vertex classes X, Y. Show that every edge of G extends to a matching from X to Y if and only if $|\Gamma(A)| > |A|$ for every $A \subset X$, $A \neq \emptyset, X$.

11. Let A be a matrix with each entry 0 or 1. Prove that the minimum number of rows and columns containing all the 1s of A equals the the maximum number of 1s that can be found with no two in the same row or column.

12. An $n \times n$ *Latin square* (resp. $r \times n$ *Latin rectangle*) is an $n \times n$ (resp. $r \times n$) matrix, with each entry from $\{1, \ldots, n\}$, such that no two entries in the same row or column are the same. Prove that every $r \times n$ Latin rectangle may be extended to an $n \times n$ Latin square.

+13. Let G be a (possibly infinite) bipartite graph, with vertex classes X, Y, such that $|\Gamma(A)| \geq |A|$ for every $A \subset X$. Give an example to show that G need not contain a matching from X to Y. Show however that if G is countable and $d(x) < \infty$ for every $x \in X$ then G does contain a matching from X to Y. Does this remain true if G is uncountable?

+14. Show that there are exactly n^{n-2} trees on a given set of n vertices.

+15. The group of all isomorphisms from a graph G to itself is called the *automorphism group* of G. Show that every finite group is the automorphism group of some graph. Is every group the automorphism group of some (possibly infinite) graph?

1. What are the 99th, 100th and 101st elements in the colex order on $\mathbb{N}^{(4)}$? For which $A \in \mathbb{N}^{(4)}$ is it true that A and the successor of A (in colex) have the same sum?

2. Let n be even, and let $\mathcal{A} \subset \mathcal{P}(X)$ be a set system that contains no chain of order 3. Prove that $|\mathcal{A}| \leq \binom{n}{n/2} + \binom{n}{n/2-1}$.

3. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an antichain not of the form $X^{(r)}$, $0 \leq r \leq n$. Must there exist a maximal chain that is disjoint from $\mathcal{A}$?

4. A set system $\mathcal{A} \subset \mathcal{P}(X)$ is called a *cross-cut* if for every $B \in \mathcal{P}(X)$ there exists $A \in \mathcal{A}$ with $B \subset A$ or $A \subset B$. Prove that every cross-cut contains a cross-cut of size at most $\binom{n}{\lfloor n/2 \rfloor}$. Does every cross-cut contain a cross-cut that is an antichain?

5. Let $x_1, \ldots, x_n$ be real numbers with $|x_i| \geq 1$ for all i, and let a be real. Show that at most $\binom{n}{\lfloor n/2 \rfloor}$ of the sums $\sum_{i \in A} x_i$, $A \subset [n]$, can lie in the open interval $(a, a+1)$.

6. A set system $\mathcal{A} \subset \mathcal{P}(X)$ is called a *down-set* if whenever $A \in \mathcal{A}$ and $B \subset A$ then also $B \in \mathcal{A}$. Show that if $\mathcal{A}$ is a (non-empty) down-set then the average size of the members of $\mathcal{A}$ is at most $n/2$.

7. Let G be a graph of order n ($n \geq 3$) with $e(G) > \binom{n}{2} - (n-2)$. Prove that G is Hamiltonian.

8. For each $r \geq 3$, construct a graph G such that G does not contain K_r but G is not $(r-1)$-partite.

9. Let G be a graph of order n that does not contain an even cycle. Prove that each vertex x of G with $d(x) \geq 3$ is a cutvertex, and deduce that G has at most $\lfloor 3(n-1)/2 \rfloor$ edges. Give (for each n) a graph for which equality holds. How does this bound compare with the maximum number of edges of a graph of order n containing no *odd* cycles?

10. A *deleted* K_r consists of a K_r from which an edge has been removed. Show that if G is a graph of order n ($n \geq r+1$) with $e(G) > e(T_{r-1}(n))$ then G contains a deleted K_{r+1}.

11. A *bowtie* consists of two triangles meeting in one vertex. Show that if G is a graph of order n ($n \geq 5$) with $e(G) > \lfloor n^2/4 \rfloor + 1$ then G contains a bowtie.

+12. Let G be an r-regular graph on $2r+1$ vertices. Prove that G is Hamiltonian.

+13. For $n = 2r+1$, give an explicit bijection $f : X^{(r)} \to X^{(r+1)}$ such that $A \subset f(A)$ for every $A \in X^{(r)}$.

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1. Show that $R(3,4) \leq 9$. By considering the graph on $\mathbb{Z}_8$ (the integers modulo 8) in which x is joined to y if $x - y = \pm 1$ or ± 2, show that $R(3,4) = 9$.

2. By considering the graph on $\mathbb{Z}_{17}$ in which x is joined to y if $x - y = \pm 1, \pm 2, \pm 4$ or ± 8, show that $R(4,4) = 18$.

3. Let $\mathcal{A} \subset [9]^{(3)}$ with $|\mathcal{A}| = 28$. How small can the lower shadow of $\mathcal{A}$ be? And the upper shadow?

4. Let $\mathcal{A} \subset X^{(r)}$, and let $U, V \subset X$ with $|U| = |V|$, $U \cap V = \emptyset$ and $\max U < \max V$. If $\mathcal{A}$ is left-compressed, can we have $|\partial C_{UV}(\mathcal{A})| > |\partial \mathcal{A}|$?

5. Find a set system $\mathcal{A}$ for which equality holds in the Kruskal-Katona theorem but which is not isomorphic to an initial segment of colex.

6. Show that every maximal intersecting family in $\mathcal{P}(X)$ has size 2^{n-1}.

7. Let A be a set of $R^{(4)}(n,5)$ points in the plane, with no three points of A collinear. Prove that A contains n points forming a convex n-gon.

8. Let $f, g_1, \ldots, g_n : \mathbb{R} \to \mathbb{R}$ be real-valued functions, with $g_1, \ldots, g_n$ bounded. Suppose that whenever $|f(x) - f(y)| \geq 1$ we have $|g_i(x) - g_i(y)| \geq 1$ for some i. Prove that f is bounded.

9. Let $\mathcal{A} \subset \mathcal{P}(X)$ be a t-intersecting family. By applying UV-compressions, for disjoint pairs U, V with $|U| > |V|$, show that if $n + t$ is even then $|\mathcal{A}| \leq \sum_{i=(n+t)/2}^{n} \binom{n}{i}$.

10. Let $\mathcal{A} \subset \mathcal{P}(X)$ be such that $|A \cap B|$ is even for all $A, B \in \mathcal{A}$, $A \neq B$. Prove that if $|A|$ is odd for all $A \in \mathcal{A}$ then $|\mathcal{A}| \leq n$, while if $|A|$ is even for all $A \in \mathcal{A}$ then $|\mathcal{A}| \leq 2^{n/2}$.

11. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{N})$ be an intersecting family of finite sets. Must there exist a finite set $F \subset \mathbb{N}$ such that the family $\{A \cap F : A \in \mathcal{A}\}$ is intersecting? And what if $\mathcal{A} \subset \mathbb{N}^{(r)}$?

12. Let the infinite subsets of $\mathbb{N}$ be 2-coloured. Must there exist an infinite set $M \subset \mathbb{N}$ all of whose infinite subsets have the same colour?

+13. Let A be an uncountable set, and let $A^{(2)}$ be 2-coloured. Must there exist an uncountable monochromatic set in A?

+14. Let $\mathcal{A} \subset X^{(r)}$ and $\mathcal{B} \subset X^{(r+1)}$ be initial segments of colex with $|\mathcal{A}| = |\mathcal{B}|$. Do we always have $|\partial \mathcal{A}| \leq |\partial \mathcal{B}|$?