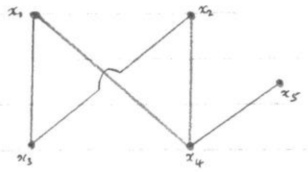
Combinatorics.

1. Graph Theory.



A graph is a pair $G=(V,E)$, where V is a set and $E \subseteq V^{(2)} = \{(x,y) \mid x,y \in V, x \neq y\}$, the set of unordered pairs from V.

Eg: in the above example, $V=\{x_1,x_2,x_3,x_4,x_5\}$, $E=\{x_1x_3, x_3x_2, x_2x_4, x_4x_1, x_4x_5\}$.

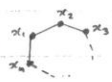
Unless otherwise stated, V is finite
Note: no loops, no multiple edges, no directed arcs.
The vertex set of G is $V=V(G)$.
The edge set of G is $E=E(G)$.
Often write $x \in G$, say, for $x \in V(G)$.
The order of G is $|G|=|V|$. The size of G is $e(G)=|E|$.

Examples: (i) the empty graph, E_n, of order n: $V=\{x_1,..,x_n\}$, $E=\emptyset$:

(ii) the complete graph, K_n, of order n: $V=\{x_1,..,x_n\}$, $E=V^{(2)}$, so $\binom{n}{2}$ edges : eg:

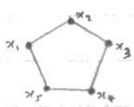
(iii) the path, P_n, of length n; $V=\{x_1,..,x_{n+1}\}$, $E=\{x_ix_{i+1} \mid 1 \le i \le n\}$, order $n+1$:

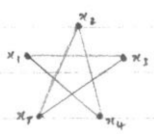
(iv) the cycle, C_n, of length $n\ (\geq 3)$: $V=\{x_1,..,x_n\}$, $E=\{x_ix_{i+1} \mid 1 \le i \le n-1\} \cup \{x_1x_n\}$:



Graphs $G=(V,E)$ and $H=(V',E')$ are isomorphic $(G \cong H)$ if $\exists$ a bijection $f: V \to V'$ such that $xy \in E \Leftrightarrow f(x)f(y) \in E'$.

Example:





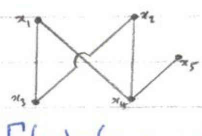
H is a subgraph of G if $V' \subset V$ and $E' \subset E$.

Example: C_n is a subgraph of K_n.

For a graph G and $xy \in E$, write $G-xy$ for $(V, E-\{xy\})$. If $xy \notin E(G)$, write $G+xy$ for $(V, E \cup \{xy\})$

If $xy \in E(G)$, say x,y are adjacent or neighbours
The neighbourhood of x is $\Gamma(x)=\{y \in V \mid xy \in E\}$.
The degree of x is $|\Gamma(x)|$, denoted by $d(x)$

Example:



$\Gamma(x_4)=(x_1,x_2,x_5)$. $\therefore d(x_4)=3$.

If $V = \{x_1, .., x_n\}$, the degree sequence of G is : $d(x_1), .., d(x_n)$

The maximum degree of G is $\Delta G = \max\{d(x_i)\}$.

The minimum degree of G is $\delta G = \min\{d(x_i)\}$.

Eg: in above example, the degree sequence is 2,2,2,3,1, so $\Delta G = 3$, $\delta G = 1$

G is regular of degree k if $d(x) = k \ \forall x \in G$.

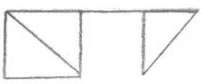
Examples: C_n is 2-regular. K_n is $(n-1)$-regular.

Trees.

G a graph, and $x, y \in V(G)$.

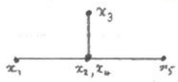
An xy-path is a sequence $x_1, .., x_k$ of distinct vertices $(k \geq 1)$ such that $x_1 = x$, $x_k = y$ and $x_i x_{i+1} \in E(G) \ \forall\ 1 \leq i \leq k-1$. It has length k.

G is connected if, $\forall x, y$, $\exists$ xy-path in G.

Example:  is connected ;  is not connected.

Note that the relation $x \sim y$ iff G has an xy-path is an equivalence relation. (Path can be a single point, so $x \sim x$).

Of course, if $x_1, .., x_k$ and $x_k, .., x_l$ are paths, then $x_1, .., x_l$ need not be a path,

(Eg. 

with x_2 and x_4 defined as the same point)

but it contains an $x_1 x_l$ path. For example, choose a minimal $1 \leq i \leq k$ such that $\exists\ j$ with $k \leq j \leq l$, with $x_j = x_i$, then $x_1 .. x_i x_j .. x_l$ is clearly an $x_1 x_l$ path

A walk in G is a sequence $x_1, .., x_k$ such that $x_i x_{i+1} \in E(G) \ \forall\ 1 \leq i \leq k-1$.

Thus, G has a path from x to y $\Leftrightarrow$ G has a walk from x to y.

A component of G is an equivalence class of $\sim$. Thus the component containing x is: $\{y \mid \exists\ xy\text{-path in } G\}$.

Example:  - two components.

G is acyclic if it does not contain a cycle.

G is a tree if it is a connected acyclic graph.

Example:  is a tree.

Proposition 1: G a graph. The following are equivalent:

(i) G is a tree, (Connected, acyclic)

(ii) G is minimal connected, (Connected, but $G-xy$ disconnected $\forall\, xy \in E(G)$).

(iii) G is maximal acyclic, (Acyclic, but $G+xy$ cyclic $\forall\, xy \notin E(G)$).

Proof: (i) $\Rightarrow$ (ii) If $G-xy$ connected, then it contains an xy-path P. But then G contains cycle Pyx ※.

(ii) $\Rightarrow$ (i) Suppose G contains a cycle C. Choose $xy \in E(C)$. Then $G-xy$ ~~conn~~ connected, as given any ab-path using xy, can replace xy with $C-xy$ to obtain an ab-walk. ※.

(i) $\Rightarrow$ (iii) For any $xy \notin E(G)$, G contains an xy path P. So $G+xy$ contains the cycle Pyx.

(iii) $\Rightarrow$ (i) Suppose G disconnected; choose x,y in different components of G. Then $G+xy$ acyclic. (contradicts maximality).

Take $x \in T$, a tree. Say x is an endvertex or leaf if $d(x)=1$.

Proposition 2: Every tree T on n vertices has $\geqslant 2$ endvertices.

Proof: Choose a path P of maximal length, say $P = x_1 \ldots x_k$. Then $\Gamma(x_1) \subset P$ (as P maximal), and $\Gamma(x_1) \cap P = \{x_2\}$ (as T acyclic). $\therefore d(x_1)=1$. Similarly, $d(x_k)=1$.

Alternatively: "Go for a walk." Suppose T has no endvertices. Choose $x_1x_2 \in E(T)$, and then define x_3, x_4 by picking $x_{k+1} \in \Gamma(x_k) - \{x_{k-1}\}$, $(k=2,3,\ldots)$. [Note that $d(x_k) \geqslant 2\ \forall k$]. This must repeat, yielding a cycle. ※. If G has one endvertex, start there.

For a graph G, $x \in V(G)$, $G-x$ denotes the graph $\left(V(G)-\{x\},\ E(G) \cap \left(V(G)-\{x\}\right)^{(2)}\right)$

Proposition 3: Every tree T on n vertices has $n-1$ edges. $(n \geqslant 1)$

Proof: Induction on n. $n=1$ trivial. Given a tree T on n vertices, choose an endvertex x. Then, $T-x$ is a tree with $|T-x| = n-1$. $\therefore e(T-x) = n-2$, so $e(T)=n-1$, as x is an endvertex.

Corollary 4: G a graph on n vertices. $(n \geqslant 1)$. Then the following are equivalent:

(i) G is a tree,

(ii) G is connected and $e(G)=n-1$.

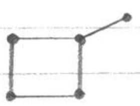
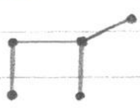
(iii) G is acyclic and $e(G)=n-1$.

Proof: (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) by definition and proposition 3.

(ii) $\Rightarrow$ (i). If G not minimal connected, remove edges to obtain minimal connected G'. Then $e(G') = n-1$. ※.

(iii) $\Rightarrow$ (i). If G not maximal acyclic, add edges to obtain maximal acyclic G'. Then $e(G') = n-1$ ※.

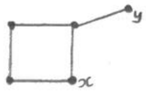
A spanning tree of a graph G is a subgraph T which is a tree with $V(T)=V(G)$.

Example:  has  as a spanning tree.

Clearly every connected G has a spanning tree. (Remove edges to obtain minimal connected).

Can use spanning trees to prove proposition 3 directly (ie. no induction).
For any connected graph G, will construct a spanning tree T with $e(T) = n-1$.
Then done, as the only spanning tree of a tree T is itself (-minimal connected).

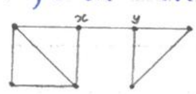
For $x,y \in G$, the distance from x to y, $d(x,y)$ = length of shortest xy-path.

Example: 

has $d(x,y) = 2$.

To construct the spanning tree: Fix $x_0 \in G$. For each $x \in V - \{x_0\}$, choose a shortest xx_0-path, say $xx' .. x_0$. So $d(x', x_0) = d(x, x_0) - 1$. Let $E(T) = \{xx' \mid x \in V - \{x_0\}\}$. So $e(T) = n-1$.
T is connected: for any $x \in V - \{x_0\}$, the sequence $x, x', x'', \dots$ reaches x_0.
T is acyclic: if T has a ~~cycle~~ cycle C, choose $x \in C$ at maximal distance from x_0, say $d(x, x_0) = k$. Then the two neighbours of x in C are at distance $\leq k$ from x_0 - ※ to construction of T (ie. only one point closer to x_0 a neighbour of x. Here we have 2)

Notes: (i) A forest is an acyclic graph. Thus, G is a forest $\Leftrightarrow$ every component of G is a tree.
Example:  is a forest.

(ii) G connected, $xy \in G$. Then xy is a bridge if $G - xy$ is disconnected.
Example: In 

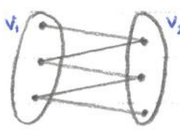
xy is a bridge.

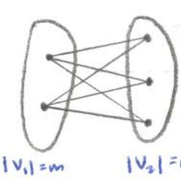
G a tree $\Leftrightarrow$ G is connected and every edge is a bridge.

(iii) A cutvertex is a vertex $x \in G$ such that $G - x$ disconnected.
Eg: x or y in the example above.
If xy is a bridge, then x is a cutvertex (unless $\Gamma(x) = \{y\}$).
Thus if G has a bridge, then G has a cutvertex (for $|G| > 2$).
The converse is false. Eg. 

Bipartite Graphs and Matchings.

A graph G is bipartite with vertex classes V_1, V_2 if V_1, V_2 partition V (ie, $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$), and $E(G) \subseteq \{xy \mid x \in V_1, y \in V_2\}$ (ie, no ~~in~~ edges inside V_1 or V_2).

Examples: (i) A path: 

(ii) The Complete Bipartite Graph K_{mn}: 

- This is K_{23}.

ie, $E(K_{mn}) = \{xy \mid x \in V_1, y \in V_2\}$.

Proposition 5: G a graph. Then, G bipartite $\Leftrightarrow$ G does not contain an odd cycle.

Notation: A *circuit* in a graph G is a closed walk, ie a walk of the form $x_1 \dots x_n x_1$.

Note that if G contains an odd circuit then G contains an odd cycle. Indeed, suppose $x_1 \dots x_k x_1$ is an odd circuit which is not a cycle. Choose $1 \le i \le j \le k$, with $x_i = x_j$. Either $x_i x_{i+1} \dots x_j$ or $x_j x_{j+1} \dots x_k x_1 \dots x_i$ is an odd circuit of shorter length. Done by induction.

Proof of proposition 5: ($\Rightarrow$) Any cycle must alternate between the two vertex classes, so even length.
($\Leftarrow$) Wlog, G connected. Fix $x_0 \in G$ and put $V_1 = \{x \mid d(x, x_0) \text{ even}\}$, $V_2 = \{x \mid d(x, x_0) \text{ odd}\}$.
If $x, y \in V_1$ or $x, y \in V_2$ and $xy \in E(G)$ (so not bipartite), then xy together with shortest path from x and y to x_0 is an odd circuit #. So bipartite.

Let G be a bipartite graph, vertex classes X, Y.
A *matching* from X to Y is a set of edges $\{xx' \mid x \in X\}$ such that $x \mapsto x'$, injective.
Equivalently, a matching is a family of $|X|$ independent edges (no two share a vertex).
When does G have a matching from X to Y?

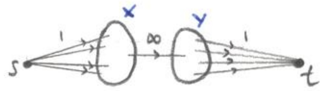
"Matchmaker terminology": $X = \{\text{boys}\}$, $Y = \{\text{girls}\}$, $x \in X$ joined to $y \in Y$ if x knows y.
Can we pair up each boy with a girl that he knows?
Certainly not if $d(x) = 0$, some $x \in X$. Similarly not if $\exists\, x_1, x_2 \in X$ ($x_1 \ne x_2$) with $\Gamma(x_1) = \Gamma(x_2) = \{y\}$, some $y \in Y$. Writing $\Gamma(A) = \bigcup_{x \in A} \Gamma(x)$, clearly need $|\Gamma(A)| \ge |A|$, $\forall\, A \subset X$.
Any more conditions? Actually, no:

Theorem 6 (Hall's (Marriage) Theorem): G bipartite, vertex classes X, Y. Then, G has a matching from X to Y $\Leftrightarrow$ $|\Gamma(A)| \ge |A|$ $\forall\, A \subset X$.

Notation: G a graph, $A \subset V(G)$. The *induced subgraph* on A is the graph $G[A] = (A, V(G) \cap A^{(2)})$.

Proof of Theorem 6: ($\Rightarrow$) Done before.
($\Leftarrow$) Induction on $|X|$. If $|X| = 1$, fine. Given bipartite G, vertex classes X, Y, with $|\Gamma(A)| \ge |A|$, $\forall\, A \subset X$, *question*: do we have $|\Gamma(A)| > |A|$ $\forall\, A \subset X$, $A \ne \emptyset, X$?
If *yes*: choose $x \in X$, $y \in \Gamma(x)$. Let G' be $G - x - y$.
Claim: G' has a matching from $X - \{x\}$ to $Y - \{y\}$.
Proof: For any $A \subset X - \{x\}$, have $|\Gamma_G(A)| > |A|$, so $|\Gamma_{G'}(A)| \ge |A|$. So true by induction.
If *no*: have $A \subset X$ ($A \ne \emptyset, X$) with $|\Gamma(A)| = |A|$. Let $G' = G[A \cup \Gamma(A)]$.
Then, $|\Gamma_{G'}(B)| \ge |B|$ $\forall\, B \subset A$ (as $\Gamma_{G'}(B) = \Gamma_G(B)$). To complete the proof, sufficient to show:
Claim: G'' has a matching from $X - A$ to $Y - \Gamma(A)$ [where $G'' = G[(X - A) \cup (Y - \Gamma(A))]$].
Proof: For any $B \subset X - A$, have $|\Gamma_G(B \cup A)| \ge |B| + |A|$.
$\therefore$ $|\Gamma_G(B \cup A) - \Gamma(A)| \ge |B|$, ie. $|\Gamma_{G''}(B)| \ge |B|$.

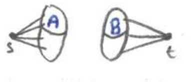
Alternative Proof: $(\Rightarrow)$ - done.

$(\Leftarrow)$ Make a directed ~~graph~~ network: 

Form directed network by directing each X-Y edge from X to Y, capacity ∞, and adding a source s, joined to each $x \in X$ by an edge of capacity 1, and a sink t, to which each $y \in Y$ is joined by an edge of capacity 1.

An integer-valued flow of capacity $|X|$ is precisely a matching from X to Y. Thus, by integral form of max-flow min-cut theorem, just need to show that each cut has capacity $\geq |X|$.

So let $A \cup B$ be a cut. $(A \subset X, B \subset Y)$. Then $\Gamma(A) \subset B$ (otherwise cut has capacity ∞).



The capacity of the cut is $\geq |B| + |X - A|$.

But $|B| \geq |\Gamma(A)| \geq |A|$, $\therefore |B| + |X-A| \geq |X|$.

G bipartite, vertex classes X, Y. A matching of deficiency d from X to Y is a set of $|X| - d$ independent edges. Eg:  has a matching of deficiency 1.

Corollary 7: G bipartite, vertex classes X, Y. Then G has a matching of deficiency d iff $|\Gamma(A)| \geq |A| - d \quad \forall A \subset X$.

Note: $d = 0$ is Hall's theorem. Corollary 7 is sometimes called "Defect Hall".

Proof: $(\Rightarrow)$ Obvious.

$(\Leftarrow)$ Form G' by adding d points to Y, each joined to all points of X.

Then, $|\Gamma_{G'}(A)| \geq |A| \quad \forall A \subset X$. $\therefore G'$ has a matching.

Remarks: (i) Or instant from max-flow min-cut.

(ii) Equivalently, $\min\{d : G \text{ has a matching of deficiency } d\} = \max\{|A| - |\Gamma(A)| : A \subset X\}$.

Let $S_1, \dots, S_n$ be sets. A transversal for $S_1, \dots, S_n$ is a sequence of distinct points $x_1, \dots, x_n$ such that $x_i \in S_i \ \forall i$.

Eg: $\{1,3,4\}, \{2,3,4\}, \{1,3\}$ has a transversal $1, 4, 3$.

Clearly, to have a transversal, we need $|\bigcup_{i \in A} S_i| \geq |A| \quad \forall A \subset \{1, \dots, n\}$.

Corollary 8: Let $S_1, \dots, S_n$ be sets. Then $S_1, \dots, S_n$ has a transversal iff $|\bigcup_{i \in A} S_i| \geq |A| \quad \forall A \subset \{1, \dots, n\}$.

Proof: Wlog, each S_i finite. Define bipartite G, vertex classes $X = \{1, \dots, n\}$, $Y = \bigcup_{i=1}^{n} S_i$, by joining $i \in X$ to $y \in Y$ if $y \in S_i$. Then we require a matching from X to Y.

For $A \subset X$, $|\bigcup_{i \in A} S_i| \geq |A|$, i.e. $|\Gamma(A)| \geq |A|$. This does $(\Leftarrow)$. $(\Rightarrow)$ is obvious.

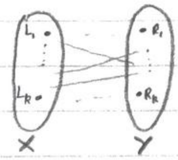
Remarks: (i) In fact, corollary 8 is equivalent to Hall: given G bipartite, on vertex classes $X = \{x_1, \dots, x_n\}$, Y, a matching is precisely a transversal for $\Gamma(x_1), \dots, \Gamma(x_n)$.

(ii) As in corollary 7, we have a defect form of corollary 8: there is a transversal for all but d of $S_1, \dots, S_n$ iff $|\bigcup_{i \in A} S_i| \geq |A| - d$, $\forall A \subset X$.

A typical application of Hall: G a finite group, H a subgroup of G. Have the left cosets $L_1, .., L_k$ of H, ($k = |G|/|H|$), each $L_i = g_i H$, and the right cosets $R_1, .., R_k$, each $R_i = Hg_i$. Can we choose a set $g_1, .., g_k$ of representatives for the left cosets and the right cosets? Ie, such that $g_1 H, .., g_k H$ are the left cosets, and $Hg_1, .., Hg_k$ are the right cosets. Ie, such that $\forall i \neq j$, $g_i^{-1} g_j \notin H$ and $g_j g_i^{-1} \notin H$.

We seek a permutation π such that $L_i \cap R_{\pi(i)} \neq \phi$ $\forall i$.

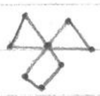
Define bipartite graph B, vertex classes $X = (L_1, .., L_k)$, $Y = (R_1, .., R_k)$, by joining L_i to R_j if $L_i \cap R_j \neq \phi$. So we want a matching from X to Y. For $A \subset \{1, .., k\}$, $|\bigcup_{i \in A} L_i| = |A| . |H|$.



$\therefore \bigcup_{i \in A} L_i$ must meet $\geq |A|$ right cosets ($|R_j| = |H|$ $\forall j$). Ie, $|\Gamma(\{L_i : i \in A\})| \geq |A|$. So done.

Euler Circuits and Hamilton Cycles

G a graph. An Euler circuit in G is a circuit passing through each edge exactly once. Ie, a circuit $x_1, .., x_k$ ($x_k = x_1$) such that $\forall xy \in E(G)$ $\exists$ unique $1 \leq i \leq k-1$ such that $xy = x_i x_{i+1}$. G is Eulerian if it has an Euler circuit.

Examples:  is Eulerian.

Any graph with a bridge is not Eulerian

Theorem 9: G a connected graph. Then G is Eulerian iff $d(x)$ is even $\forall x \in G$.

Remark: Thus for any graph G, G Eulerian iff $d(x)$ is even and at most one component of G contains an edge.

Proof of 9: ($\Rightarrow$) If an Euler circuit passes through x k times, then $d(x) = 2k$.

($\Leftarrow$) Induction on $e(G)$. If $e(G) = 0$, done.

Given connected G, $d(x)$ even $\forall x \in G$, $e(G) > 0$. Suppose G is not Eulerian. Let C be a longest circuit without repeated edges. Note that C has length > 0, as G has a cycle (as $d(x) \geq 2$ $\forall x \in G$).

Let $G' = G - E(C)$, ie, $G' = (V(G), E(G) - E(C))$, and let H be a component of G' with $e(H) > 0$. Have $V(H) \cap V(C) \neq \phi$, as G is connected. So H is connected and $d_H(x)$ is even $\forall x \in V(H)$, (because $d_C(x)$ even $\forall x \in V(C)$).

Thus H has an Euler circuit C' (hypothesis).

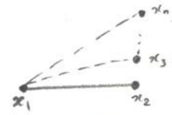
But C, C' are edge-disjoint and have a vertex in common. Hence, can combine C and C' to obtain a circuit longer than C, with no repeated edge ※.

G a graph of order n. A Hamilton cycle in G is a cycle of length n. (Ie, a cycle passing through all vertices of G). G is Hamiltonian if it has a Hamilton cycle.

Examples:  is Hamiltonian

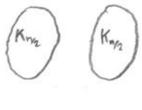
Any G with a cutvertex is not Hamiltonian.

No nice $\Leftrightarrow$ condition known for Hamiltoneity.
How "large" must a graph of order n be to ensure a Hamilton cycle?
Silly question: How many edges ensure G Hamiltonian?

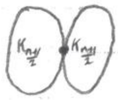


If $d(x)=1$, some $x \in G$, then G is not Hamiltonian. Then can have $e(G) = \binom{n}{2} - (n-2)$ without G being Hamiltonian. Eg, let G be the complement of $\{x_1x_2, \dots, x_1x_{n-1}\}$. [Note: complement of $G=(V,E)$ is $\bar{G} = (V, V^{(2)} - E)$].

Better question: What minimum degree guarantees G Hamiltonian?

n even: 

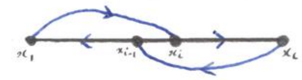
G = disjoint union of two $K_{\frac{n}{2}}$'s.
Then, $\delta(G) = \frac{n}{2} - 1$, but G not Hamiltonian.

n odd: 

G = two $K_{\frac{n+1}{2}}$'s meeting at a point.
Then, $\delta(G) = \frac{n-1}{2}$, but G not Hamiltonian.

Theorem 10: G a graph of order ≥ 3. Then, $\delta(G) \geq \frac{n}{2} \Rightarrow G$ Hamiltonian.

Proof: Note that G connected. Indeed, if x,y non-adjacent, then $\Gamma(x), \Gamma(y) \subset V - \{x,y\}$, ie. $\Gamma(x) \cap \Gamma(y) \neq \emptyset$.
Let P be a path in G of maximal length, say $P = x_1 \dots x_l$. Have $l \geq 3$ as G connected, $|G| \geq 3$. Wlog, G contains no cycle of length l. Because: if $l=n$, done, and if $l<n$ and C is an l-cycle, then, as G connected, $\exists x \in V(G) - V(C)$ adjacent to C which yields a path on $l+1$ points.



Thus $x_1x_l \notin E$. Moreover, for each $2 \leq i \leq l$, we cannot have $x_1x_i, x_{i-1}x_l \in E$ (else G has an l-cycle).
But $\Gamma(x_1) \subset \{x_2, \dots, x_l\}$ and $\Gamma(x_l) \subset \{x_1, \dots, x_{l-1}\}$.
So $\Gamma(x_1)$ and $\Gamma_+(x_l) = \{x_i : x_{i-1} \in \Gamma(x_l)\}$ are disjoint subsets of $\{x_2, \dots, x_l\}$ #.

Remark: We did not make full use of $\delta(G) \geq \frac{n}{2}$. We could replace this by: x,y non-adjacent $\Rightarrow d(x) + d(y) \geq n$.

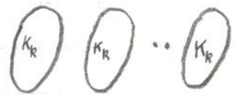
Theorem 10': G a graph of order $n \geq 3$. Then $\delta(G) \geq \frac{k}{2}$ $(k<n)$, G connected $\Rightarrow G$ has a path of length k.

Note: Need $k<n$, eg. $G = K_k$
Also need G connected - eg G the disjoint union of two K_k's.

Proof: Choose a path $P = x_1 \dots x_l$ of maximal length and suppose $l \leq k$.
Have $l \geq 3$ as G connected, $|G| \geq 3$. Then G contains no l-cycle, as in Theorem 10.
So $\Gamma(x_1)$ and $\Gamma_+(x_l) = \{x_i : x_{i-1}x_l \in E\}$ are disjoint subsets of $\{x_2, \dots, x_l\}$, as in Theorem 10, contradicting $|\Gamma(x_1)|, |\Gamma(x_l)| \geq k/2$.

Theorem 11: G a graph of order n containing no path of length k. Then $e(G) \leq \frac{k-1}{2}n$.

Remark: Cannot improve this if n is multiple of k. Eg, G a disjoint union of $\frac{n}{k}$ K_k's has no no path of length k, but $e(G) = \frac{k-1}{2}n$



Proof: Induction on n. $n \leq 2$, okay.
Given G of order $n \geqslant 3$, containing no path of length k.
Wlog, G connected. If ~~let~~ not, let components be $G_1, .., G_r$, orders $n_1, .., n_r$.
Then, $e(G_i) \leq \frac{k-1}{2}n_i$ (induction), so $e(G) \leq \frac{k-1}{2}n$. Done.
By Theorem 10', G has a vertex x with $d(x) \leq \frac{k-1}{2}$. Then $G-x$ is a graph of order $n-1$, containing no path of length k.
Hence, $e(G-x) \leq (\frac{k-1}{2})(n-1)$. So $e(G) \leq (\frac{k-1}{2})(n-1) + d(x) \leq \frac{k-1}{2}n$.

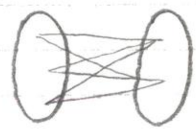
Remark: Theorems 10 and 11 are extremal problems: how "large" can a graph of order n be with a certain property? Often this property is non-containment of a certain subgraph. (eg. C_n in Theorem 10, P_n in Theorem 11).

How many edges guarantees a K_k?

Turan's Theorem

G a graph of order n, $G \not\supset K_r$. How large can $e(G)$ be?

Eg: $r=3$. Obvious choice is complete bipartite $K_{a,b}$ $(a+b=n)$
Best to take $a=b=\frac{n}{2}$ (n even), $a=\frac{n+1}{2}$, $b=\frac{n-1}{2}$ (n odd)

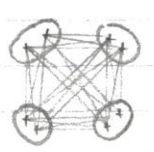


A graph G is k-partite, with vertex classes $V_1, .., V_k$ if $V_1, .., V_k$ partitions V and $G[V_i]$ is empty $\forall i$.
G is complete k-partite, vertex classes $V_1, .., V_k$, if $V_1, .., V_k$ partition $V(G)$ and $E(G) = \{xy : x \in V_i, y \in V_j, \text{ some } i \neq j\}$.
Clearly, G $(r-1)$-partite $\Rightarrow$ $G \not\supset K_r$ (else some V_i contains $\geqslant 2$ points of the K_r. ※).

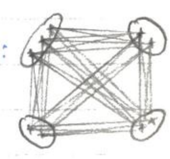
If n is a multiple of $(r-1)$, we'd try G to be complete $(r-1)$-partite, each vertex class of size $\frac{n}{r-1}$. In general, we'd take G complete $(r-1)$-partite, vertex classes as equal as possible. (Integers $a_1, .., a_k$ are as equal as possible if $|a_i - a_j| \leq 1$ $\forall i,j$)

The Turan Graph: $T_k(n)$, complete k-partite, vertex classes $V_1, .., V_k$, where $\sum |V_i| = n$ and $|V_1|, .., |V_k|$ are as equal as possible.

Eg: $T_4(8)$:



$T_4(9)$:



Clearly, $T_{r-1}(n)$ contains no K_r. Also, it is maximal K_r-free: the addition of any edge gives a K_r. Each class of $T_{r-1}(n)$ has size $\lceil \frac{n}{r-1} \rceil$ or $\lfloor \frac{n}{r-1} \rfloor$.
Also, $x \in V_i$ has $\Gamma(x) = \bigcup_{j \neq i} V_j$, so $d(x) = n - |V_i|$.

Theorem 12 (Turan's Theorem): G a graph of order n, $e(G) > e(T_{r-1}(n))$. Then $G \supset K_r$.

Key idea: Strengthen the statement to make it easier to prove.

Proof: will show that $|G| = n$, $e(G) = e(T_{r-1}(n))$, $G \not\supset K_r \Rightarrow G \cong T_{r-1}(n)$.

[This certainly implies Turan by the maximality of $T_{r-1}(n)$].

Prove by induction on n. If $n = r-1$, done.

Given G, $|G| = n \geq r$, $e(G) = e(T_{r-1}(n))$, $G \not\supset K_r$.

Claim: $\delta(G) \leq \delta(T_{r-1}(n))$.

Proof: Have $\sum_{x \in G} d(x) = \sum_{x \in T_{r-1}(n)} d(x)$. But the $d(x) : x \in T_{r-1}(n)$ are as equal as possible.

$\therefore \min\{d(x) : x \in G\} \leq \min\{d(x) : x \in T_{r-1}(n)\}$.

Choose $x \in G$ with $d(x) = \delta(G)$, and let $G' = G - x$. Then $|G'| = n-1$ and $G' \not\supset K_r$.

Also, $e(G') = e(G) - \delta(G) \geq e(T_{r-1}(n)) - \delta(T_{r-1}(n)) = e(T_{r-1}(n-1))$.

[To obtain $T_{r-1}(n-1)$ from $T_{r-1}(n)$, remove a vertex in a largest V_i].

Thus, $G' \cong T_{r-1}(n-1)$ by induction and $d(x) = \delta(T_{r-1}(n))$.

Let G' have vertex classes $V_1, \dots, V_{r-1}$ (all non-empty, as $n-1 \geq r-1$). Cannot have x joined to a point in each V_i (else $G \supset K_r$). $\therefore \Gamma(x) \cap V_i = \phi$, some V_i.

But $|\Gamma(x)| = n-1 - \min_i |V_i|$. [To obtain $T_{r-1}(n)$ from $T_{r-1}(n-1)$, add a point to a smallest V_i].

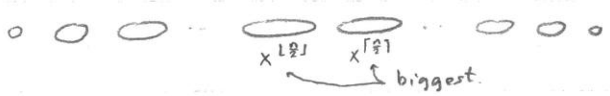
Hence, $\Gamma(x) = \bigcup_{j \neq i} V_j$, some i with $|V_i|$ minimal. Thus G is complete $(r-1)$-partite, vertex classes $\{V_j : j \neq i\}$ and $V_i \cup \{x\}$. So $G \cong T_{r-1}(n)$.

2. Set Systems.

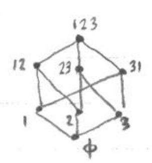
Let X be a set. A set system on X, or a family of subsets of X is a set $\mathcal{A} \subset \mathbb{P}(X)$.
Usually, $X = [n] = \{1, \dots, n\}$, for $n = 1, 2, 3, \dots$

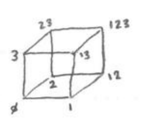
Example: $X^{(r)} = \{A \subset X : |A| = r\}$. Thus $|X^{(r)}| = \binom{n}{r}$.

n even: 

n odd: 

Often view $\mathbb{P}(X)$ as a graph, by joining A to B if $A = B \cup \{i\}$, some $i \notin B$, or vice versa. Thus, A joined to $B \Leftrightarrow |A \triangle B| = 1$. This graph is called Q_n, the discrete cube.

Example: Q_3: 

or: 

[Regard $A \in \mathbb{P}(X)$ as the point $(a_1, \dots, a_n) \in \mathbb{R}^n$, where $a_i = 1$ if $i \in A$, 0 if $i \notin A$]

Chains and Antichains.

$\mathcal{A} \subset \mathbb{P}(X)$ is a chain if $\forall A, B \in \mathcal{A}$ we have $A \subset B$ or $B \subset A$.
$\mathcal{A} \subset \mathbb{P}(X)$ is an antichain, or Sperner system, if $\forall A, B \in \mathcal{A}$, $A \neq B$ we have $A \not\subset B$.

Clearly, no chain has size $> n+1$. [$\mathcal{A}$ a chain $\Rightarrow |\mathcal{A} \cap X^{(r)}| = 1 \ \forall r$]
How large can an antichain be? Eg, $X^{(r)}$ an antichain for any r. Can we beat $X^{(\lfloor \frac{n}{2} \rfloor)}$?

Theorem 1 (Sperner's Lemma): Let $\mathcal{A}$ be an antichain in $\mathbb{P}(X)$. Then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$

Proof: Enough to decompose $\mathbb{P}(X)$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains. For this, enough to show that:
(i) $\forall r < n/2$, $\exists$ injection $f: X^{(r)} \to X^{(r+1)}$ such that $A \subset f(A) \ \forall A \in X^{(r)}$, and
(ii) $\forall r > n/2$, $\exists$ injection $f: X^{(r)} \to X^{(r-1)}$ such that $A \supset f(A) \ \forall A \in X^{(r)}$.
[These fit together to form $\binom{n}{\lfloor n/2 \rfloor}$ chains - each one meets the middle layer in exactly one point].
By taking complements, enough to prove (i)
Consider the bipartite graph on vertex classes $X^{(r)}$ and $X^{(r+1)}$, in which $A \in X^{(r)}$ is joined to $B \in X^{(r+1)}$ if $A \subset B$ (ie. $Q_n[X^{(r)} \cup X^{(r+1)}]$). We seek a matching from $X^{(r)}$ to $X^{(r+1)}$. By Hall, sufficient to check that $|\Gamma(S)| \geq |S| \ \forall S \subset X^{(r)}$.
For $S \subset X^{(r)}$, how many edges from S to $\Gamma(S)$? # of S-$\Gamma(S)$ edges $= |S|(n-r)$, (counting from S)
and $\leq |\Gamma(S)|(r+1)$, (counting from $\Gamma(S)$).
Hence, $|S|(n-r) \leq |\Gamma(S)|(r+1)$, so $|\Gamma(S)| \geq |S| . \frac{n-r}{r+1} \geq |S|$ (as $r+1 \leq n-r$)

Remarks: (i) Result is best possible, eg. $\mathcal{A} = X^{(\lfloor \frac{n}{2} \rfloor)}$.
(ii) The proof says nothing about uniqueness of maximum-sized antichains.

Aim: $\mathcal{A}$ an antichain $\Rightarrow \sum_{r=0}^{n} \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1$. (Clearly implies Sperner).

For $\mathcal{A} \subset X^{(r)}$ $(1 \leq r \leq n)$ the shadow or lower shadow of $\mathcal{A}$ is $\partial \mathcal{A} = \delta \mathcal{A} = \{B \in X^{(r-1)} : B \subset A, \text{ some } A \in \mathcal{A}\}$

Eg: $\partial\{123, 235\} = \{12, 13, 23, 25, 35\}$

Lemma 2 (Local LYM): Let $\mathcal{A} \subset X^{(r)}$, $1 \leq r \leq n$. Then $\frac{|\partial \mathcal{A}|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}$
"The shadow occupies a greater fraction of the layer below"
Proof: Consider the sub-graph of Q_n spanned by $X^{(r)} \cup X^{(r-1)}$.
of $\mathcal{A}$-$\partial\mathcal{A}$ edges $= |\mathcal{A}| r$, (counting from $\mathcal{A}$)
and, $\leq |\partial \mathcal{A}|(n-r+1)$, (counting from $\partial \mathcal{A}$).
Then, $\frac{|\partial \mathcal{A}|}{|\mathcal{A}|} \geq \frac{r}{n-r+1}$. But, $\frac{\binom{n}{r-1}}{\binom{n}{r}} = \frac{r!\,(n-r)!}{(r-1)!\,(n-r+1)!} = \frac{r}{n-r+1}$.

Remark: Inequality is strict if $\mathbb{P}(\partial \mathcal{A}) \not\subset \mathcal{A}$. So, equality $\Leftrightarrow \mathcal{A} = \emptyset$ or $X^{(r)}$ (as the graph is connected)

Theorem 3 (LYM inequality): $\mathcal{A} \subset \mathbb{P}(X)$ an antichain $\Rightarrow \sum \frac{|\mathcal{A} \cap X^{(r)}|}{\binom{n}{r}} \leq 1$. [LYM: Lubell, Yamamoto, Meshalkin].
Proof: Write $\mathcal{A}_r$ for $\mathcal{A} \cap X^{(r)}$ $(0 \leq r \leq n)$.
Sets $\partial \mathcal{A}_n$ and $\mathcal{A}_{n-1}$ are disjoint $\Rightarrow \frac{|\partial \mathcal{A}_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} = \frac{|\partial \mathcal{A}_n|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} \geq \frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}}$ (by local LYM).
Now, the sets $\partial^2 \mathcal{A}_n \cup \partial \mathcal{A}_{n-1}$ and $\mathcal{A}_{n-2}$ are disjoint.
$\therefore \frac{|\partial^2 \mathcal{A}_n \cup \partial \mathcal{A}_{n-1} \cup \mathcal{A}_{n-2}|}{\binom{n}{n-2}} = \frac{|\partial^2 \mathcal{A}_n \cup \partial \mathcal{A}_{n-1}|}{\binom{n}{n-2}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \geq \frac{|\partial \mathcal{A}_n \cup \mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}} \geq \frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \frac{|\mathcal{A}_{n-2}|}{\binom{n}{n-2}}$

Continue inductively. We obtain: $\frac{|\partial^n \mathcal{A}_n \cup \partial^{n-1} \mathcal{A}_{n-1} \cup \dots \cup \partial \mathcal{A}_1 \cup \mathcal{A}_0|}{\binom{n}{0}} \geq \frac{|\mathcal{A}_n|}{\binom{n}{n}} + \frac{|\mathcal{A}_{n-1}|}{\binom{n}{n-1}} + \dots + \frac{|\mathcal{A}_1|}{\binom{n}{1}} + \frac{|\mathcal{A}_0|}{\binom{n}{0}}$
But LHS ≤ 1.

Remark: If we have equality in LYM, then we had equality each time we applied local LYM. Hence the first time $\mathcal{A}_r \neq \emptyset$ (ie, maximal such r), have $\mathcal{A}_r = X^{(r)}$. So $\mathcal{A} = X^{(r)}$.
In particular, equality in Sperner $\Leftrightarrow \mathcal{A} = X^{(\lfloor n/2 \rfloor)}$ or $\mathcal{A} = X^{(\lceil n/2 \rceil)}$.

Proof 2 of LYM: Choose a maximal chain $\mathcal{C}$ at random. (Ie. $A_0 \subset A_1 \subset \dots \subset A_n$, with $|A_i| = i \ \forall i$)
For a given r-set A, $P(A \in \mathcal{C}) = \frac{1}{\binom{n}{r}}$ (as all r-sets equally likely).
So $P(\mathcal{C} \text{ meets } \mathcal{A}_r) = \frac{|\mathcal{A}_r|}{\binom{n}{r}}$ (events are disjoint).

$\therefore P(\mathcal{C} \text{ meets } \mathcal{A}) = \sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}}$ (events are disjoint). But LHS ≤ 1.

Remark: Of course, can rephrase without probability:
maximal chains $= n!$
maximal chains containing a given r-set $A = r!(n-r)!$ (go from $\emptyset$ to A, then from A to X).
So, # maximal chains meeting $\mathcal{A}$ is $\sum_{r=0}^{n} |\mathcal{A}_r| \, r! (n-r)!$, which is $\leq n!$

Shadows.

For $\mathcal{A} \subset X^{(r)}$, local LYM tells us that $|\partial \mathcal{A}| \geq |\mathcal{A}| \frac{r}{n-r+1}$, but with equality only for $\mathcal{A} = \emptyset$ or $\mathcal{A} = X^{(r)}$. Ie, not often. How small can $\partial\mathcal{A}$ be, given $\mathcal{A}$? In other words, how tightly can we 'pack' a given number of r-sets?

Examples: $r=2, n=5, |\mathcal{A}|=3$. $\mathcal{A} = \{12, 13, 14\} \rightarrow \partial \mathcal{A} = \{1,2,3,4\}$. : $|\partial\mathcal{A}| = 4$
$\mathcal{A} = \{12, 13, 23\} \rightarrow \partial\mathcal{A} = \{1,2,3\}$: $|\partial \mathcal{A}| = 3$.

$r=3, n=7, |\mathcal{A}|=4$. $\mathcal{A} = \{123, 124, 125, 126\} \rightarrow \partial\mathcal{A} = \{12, 13, 23, 14, 24, 15, 25, 16, 26\}$: $|\partial\mathcal{A}| = 9$.
$\mathcal{A} = \{123, 124, 134, 234\} \rightarrow \partial\mathcal{A} = [4]^{(2)}$: $|\partial\mathcal{A}| = 6$.

This suggests that $|\mathcal{A}| = \binom{k}{r} \Rightarrow |\partial\mathcal{A}| \geq \binom{k}{r-1}$. [or if $\mathcal{A} = [k]^{(r)}$, then $\partial\mathcal{A} = [k]^{(r-1)}$].
What if $\binom{k}{r} < |\mathcal{A}| < \binom{k+1}{r}$, some k? Might take $[k]^{(r)}$, together with a subset of $[k+1]^{(r)}$.

Example: $r=3$, $|\mathcal{A}| = \binom{7}{3} + \binom{4}{2}$. Try $\mathcal{A} = [7]^{3} \cup \{A \cup \{8\} : A \in [4]^{(2)}\}$.

Two important total orderings of $X^{(r)}$.

Lex: Let $A, B \in X^{(r)}$: $A = \{a_1, \dots, a_r\}$, $a_1 < \dots < a_r$; $B = \{b_1, \dots, b_r\}$, $b_1 < \dots < b_r$.
Then $A < B$ in the lexicographic order if $a_1 < b_1$, or $a_1 = b_1$ and $a_2 < b_2$, or $a_1 = b_1, \dots, a_{r-1} = b_{r-1}, a_r < b_r$.
So, $A < B \Leftrightarrow a_s < b_s$, where $s = \min\{t : a_t \neq b_t\}$.

Examples: $[4]^{(2)}$: 12, 13, 14, 23, 24, 34.
$[n]^{(3)}$: 123, 124, .., 12n, 134, 135, .., 13n, .., 1(n-1)n, 234, 235, .., 23n, 245, .., 24n, ..., (n-2)(n-1)n.

Colex: $A<B$ in the colexicographic order if $a_r<b_r$, or $a_r=b_r$ and $a_{r-1}<b_{r-1}$, or $a_r=b_r, a_{r-1}=b_{r-1}, \dots, a_2=b_2$ and $a_1<b_1$.
So $A<B \iff a_s<b_s$ where $s=\max\{t: a_t\neq b_t\}$.
Equivalently, $A<B$ if $\sum_{i\in A}2^i < \sum_{i\in B}2^i$.

Examples: $[4]^{(2)}$: 12, 13, 23, 14, 24, 34
$[n]^{(3)}$: 123, 124, 134, 234, 125, 135, 235, 145, 245, 345, 126, ...
In $[n]^{(4)}$, successor of 2458 is 3458, then 1268, etc.

Note that $[n]^{(r)}$ is an initial segment of $[n+1]^{(r)}$ in colex (initial segment = first m points, some m), although not in lex. So could view colex as an enumeration of $\mathbb{N}^{(r)}$.

Aim: To show that initial segments of colex have smallest shadow, ie. if $\mathcal{A}\subset X^{(r)}$ and $\mathcal{C}$ is the first $|\mathcal{A}|$ points of $X^{(r)}$ in colex, then $|\partial\mathcal{C}|\leq|\partial\mathcal{A}|$.
In particular, $|\mathcal{A}|=\binom{k}{r} \Rightarrow |\partial\mathcal{A}|\geq\binom{k}{r-1}$.

Compressions.

Try to replace $\mathcal{A}$ by $\mathcal{A}'$, where: (i) $|\mathcal{A}'|=|\mathcal{A}|$,
(ii) $|\partial\mathcal{A}'|\leq|\partial\mathcal{A}|$
(iii) $\mathcal{A}'$ "looks more like" $\mathcal{C}$ than $\mathcal{A}$ did.
(Important not just that (ii) is true, but that we can prove it).
Ideally, we would have $\mathcal{A}\to\mathcal{A}'\to\mathcal{A}''\to\dots\to\mathcal{B}$, with $\mathcal{B}$ so close to $\mathcal{C}$ that one clearly has $|\partial\mathcal{B}|\geq|\partial\mathcal{C}|$.

ij-compressions: Let $1\leq i\leq j\leq n$. For $A\in X^{(r)}$, define $C_{ij}(A)=\begin{cases} A-\{j\}\cup\{i\} & \text{if } j\in A, i\notin A.\\ A & \text{otherwise.}\end{cases}$
For $\mathcal{A}\subset X^{(r)}$, the ij-compression of $\mathcal{A}$ is the system $C_{ij}(\mathcal{A})\subset X^{(r)}$ defined by
$C_{ij}(\mathcal{A})=\{C_{ij}(A): A\in\mathcal{A}\}\cup\{A\in\mathcal{A}: C_{ij}(A)\in\mathcal{A}\}$.

Example: $C_{12}\{125, 146, 156, 256, 257, 367\}=\{125, 146, 156, 256, 157, 367\}$.
Certainly $|C_{ij}(\mathcal{A})|=|\mathcal{A}|$

Lemma 4: Let $1\leq i\leq j\leq n$, and $\mathcal{A}\subset X^{(r)}$. Then $|\partial C_{ij}(\mathcal{A})|\leq|\partial\mathcal{A}|$.
Proof: Write $\mathcal{A}'$ for $C_{ij}(\mathcal{A})$. To show $|\partial\mathcal{A}'|\leq|\partial\mathcal{A}|$, will show: (i) if $i,j\in B$, some $B\in\partial\mathcal{A}'$, then $B\in\partial\mathcal{A}$.
(ii) if $i,j\notin B$, some $B\in\partial\mathcal{A}'$, then $B\in\partial\mathcal{A}$.
(iii) for $B\in X^{(r-2)}$, $i,j\notin B$, $|\partial\mathcal{A}\cap\{B\cup\{i\},B\cup\{j\}\}|\geq|\partial\mathcal{A}'\cap\{B\cup\{i\},B\cup\{j\}\}|$.

(i): have $B\cup\{x\}\in\mathcal{A}'$, some x. But $i,j\in B\cup\{x\}$ $\therefore$ $B\cup\{x\}\in\mathcal{A}$ $\therefore$ $B\in\partial\mathcal{A}$.
(ii): have $B\cup\{x\}\in\mathcal{A}'$, some x. If $x\neq i,j$, then $i,j\notin B\cup\{x\}$, so $B\cup\{x\}\in\mathcal{A}$.
If $x=i$ or j, have $B\cup\{i\}$ or $B\cup\{j\}\in\mathcal{A}'$, so $B\cup\{i\}$ or $B\cup\{j\}\in\mathcal{A}$.
(iii): Will show: (iv) if $B\cup\{i\}\in\partial\mathcal{A}'$, then $B\cup\{i\}$ or $B\cup\{j\}\in\partial\mathcal{A}$.
(v) if $B\cup\{j\}\in\partial\mathcal{A}'$, then $B\cup\{i\}, B\cup\{j\}\in\partial\mathcal{A}$.

(iv): have $B \cup \{i,x\} \in \mathcal{A}'$, some x. If $x=j$, $B\cup\{i,j\} \in \mathcal{A}'$, so $B \cup\{i,j\}\in\mathcal{A}$, so $B\cup\{i\}, B\cup\{j\} \in \partial\mathcal{A}$.
If $x \neq j$, have $B\cup\{i,x\}$ or $B\cup\{j,x\} \in \mathcal{A}$.

(v): have $B\cup\{j,x\} \in \mathcal{A}'$, some x. If $x=i$, $B\cup\{i,j\}\in\mathcal{A}'$, so $B\cup\{i,j\}\in\mathcal{A}$.
If $x\neq i$, must have $B\cup\{j,x\}$ and $B\cup\{i,x\}\in\mathcal{A}$.

Remark: We have actually shown that $\partial C_{ij}(\mathcal{A}) \subset C_{ij}(\partial\mathcal{A})$.

For $1\le i<j\le n$ and $\mathcal{A}\subset X^{(r)}$, say $\mathcal{A}$ is ij-compressed if $C_{ij}(\mathcal{A})=\mathcal{A}$. Say $\mathcal{A}$ is left-compressed if $\mathcal{A}$ is ij-compressed $\forall\, i<j$ (ie, if $A\in\mathcal{A}$ has $i\notin A$, $j\in A$, some $i<j$, then also $A\cup\{i\}-\{j\}\in\mathcal{A}$).

Corollary 5: Let $\mathcal{A}\subset X^{(r)}$. Then there is a left-compressed $\mathcal{B}\subset X^{(r)}$ such that $|\mathcal{B}|=|\mathcal{A}|$ and $|\partial\mathcal{B}|\le|\partial\mathcal{A}|$

Proof: Define a sequence $\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2,\dots\subset X^{(r)}$ as follows. Set $\mathcal{A}_0=\mathcal{A}$. Having defined $\mathcal{A}_0,\dots,\mathcal{A}_k$, if $\mathcal{A}_k$ is left-compressed then stop the sequence with $\mathcal{A}_k$. Otherwise, choose $i<j$ such that $\mathcal{A}_k$ is not ij-compressed and set $\mathcal{A}_{k+1}=C_{ij}(\mathcal{A}_k)$ and continue inductively. This must terminate, or otherwise $\sum_{A\in\mathcal{A}_k}\sum_{i\in A} i > \sum_{A\in\mathcal{A}_{k+1}}\sum_{i\in A} i \;\; \forall k$ ✱.
The left-compressed $\mathcal{A}_k$ satisfies $|\mathcal{A}_k|=|\mathcal{A}|$ and $|\partial\mathcal{A}_k|\le|\partial\mathcal{A}|$, by Lemma 4.

Remarks: (i) Or: among all $\mathcal{B}\subset X^{(r)}$ with $|\mathcal{B}|=|\mathcal{A}|$ and $|\partial\mathcal{B}|\le|\partial\mathcal{A}|$, choose one with minimal $\sum_{A\in\mathcal{B}}\sum_{i\in A} i$. Then $C_{ij}(\mathcal{B})=\mathcal{B}$ $\forall\, i,j$.

(ii) If we want to use each C_{ij} only once, then order is important.
Eg: $\{34,35\}$ is 12-compressed, but $C_{23}\{34,35\}=\{24,25\}$ is not 12-compressed.

(iii) Clearly, any initial set of colex is left-compressed. But converse is certainly not true. Eg: $\{123,124,125,126\}$.

"Colex prefers 23 to 14" inspires the following.
Let $U,V\in\mathbb{P}(X)$, with $U\cap V=\emptyset$ and $|U|=|V|$. For $A\in X^{(r)}$, define $C_{UV}(A)=\begin{cases} A\cup U-V & \text{if } V\subset A,\ U\cap A=\emptyset \\ A & \text{otherwise.}\end{cases}$
For $\mathcal{A}\subset X^{(r)}$, define $C_{UV}(\mathcal{A})=\{C_{UV}(A): A\in\mathcal{A}\}\cup\{A\in\mathcal{A}: C_{UV}(A)\in\mathcal{A}\}$.

Example: $C_{23,14}\{123,125,146,148,236,237,359\}=\{123,125,146,238,236,237,359\}$.

Note: we have not redefined $C_{ij}=C_{\{i\},\{j\}}$.

Clearly, $|C_{UV}(\mathcal{A})|=|\mathcal{A}|$. Need not have $|\partial C_{UV}(\mathcal{A})|\le|\partial\mathcal{A}|$. For example, $\mathcal{A}=\{14,15\}$, $\partial\mathcal{A}=\{1,4,5\}$, but $\mathcal{A}'=C_{23,14}(\mathcal{A})=\{23,15\}$ has $\partial\mathcal{A}'=\{1,2,3,5\}$. Say $\mathcal{A}$ is UV-compressed if $C_{UV}(\mathcal{A})=\mathcal{A}$.

Lemma 6: Let $U,V\in\mathbb{P}(X)$ with $|U|=|V|$ and $U\cap V=\emptyset$, and let $\mathcal{A}\subset X^{(r)}$. Suppose that ⊛: $\forall u\in U, \exists v\in V$ such that $\mathcal{A}$ is $U-\{u\}, V-\{v\}$-compressed. Then $|\partial C_{UV}(\mathcal{A})|\le|\partial\mathcal{A}|$

Proof: Write $\mathcal{A}'$ for $C_{UV}(\mathcal{A})$. Let $B\in\partial\mathcal{A}'-\partial\mathcal{A}$.

Claim: $U\subset B$, $V\cap B=\emptyset$ and $B\cup V-U\in\partial\mathcal{A}-\partial\mathcal{A}'$ [Then done, as $|\partial\mathcal{A}'-\partial\mathcal{A}|\le|\partial\mathcal{A}-\partial\mathcal{A}'|$]

Proof: Have $B\in\partial\mathcal{A}'-\partial\mathcal{A}$. So $B\cup\{x\}\in\mathcal{A}'-\mathcal{A}$, some x. Thus $U\subset B\cup\{x\}$, $V\cap(B\cup\{x\})=\emptyset$ and $B\cup\{x\}\cup V-U\in\mathcal{A}$.

$V\cap B=\emptyset$: We know $V\cap(B\cup\{x\})=\emptyset$, so done.

$U\subset B$: We know $U\subset B\cup\{x\}$. If $x\notin U$, done. If $x\in U$, we know $B\cup\{x\}\cup V-U\in\mathcal{A}$ and $\mathcal{A}$ is $U-\{x\}, V-\{y\}$ compressed, some $y\in V$. Thus $B\cup\{y\}\in\mathcal{A}$, so $B\in\partial\mathcal{A}$. ✱ So done.

$B \cup V - U \in \partial A$: Have $B \cup \{x\} \cup V - U \in \mathcal{A}$, so done.

$B \cup V - U \notin \partial \mathcal{A}'$: Suppose not, so $B \cup \{w\} \cup V - U \in \mathcal{A}'$, some w. So also, $B \cup \{w\} \cup V - U \in \mathcal{A}$, (as it contains V). If $w \in U$, then $\mathcal{A}$ is $U - \{w\}, V - \{z\}$ compressed, for some $z \in V$, so $B \cup \{z\} \in \mathcal{A}$. ✱. So $w \notin U$. Thus the set $B \cup \{w\} \cup V - U$ contains V, misses U, and belongs to $\mathcal{A}'$. $\therefore$ Both $B \cup \{w\} \cup V - U$ and $C_{UV}(B \cup \{w\} \cup V - U) = B \cup \{w\}$ belong to $\mathcal{A}$. #. Done.

Remark: Can check $\partial(C_{UV}(\mathcal{A})) \subset C_{UV}(\partial \mathcal{A})$.

Theorem 7 (Kruskal-Katona Theorem): Let $1 \le r \le n$ and $\mathcal{A} \subset X^{(r)}$. Let $\mathcal{C}$ be the set of the first $|\mathcal{A}|$ elements of $X^{(r)}$ in colex. Then $|\partial \mathcal{A}| \ge |\partial \mathcal{C}|$. In particular, if $|\mathcal{A}| = \binom{k}{r}$ then $|\partial \mathcal{A}| \ge \binom{k}{r-1}$.

Proof: Let $\Gamma = \{(U,V) : U, V \in \mathbb{P}(X),\ U \cap V = \phi,\ |U| = |V| > 0,\ \max U < \max V\}$. Define a sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \dots$ as follows. Set $\mathcal{A}_0 = \mathcal{A}$. Having defined $\mathcal{A}_0, \dots, \mathcal{A}_k$, if $\mathcal{A}_k$ is UV-compressed $\forall\ (U,V) \in \Gamma$ then stop the sequence with $\mathcal{A}_k$. Otherwise, choose $(U,V) \in \Gamma$, $|U|$ minimal, with $\mathcal{A}_k$ not UV-compressed. Note that $\forall u \in U$ we have $(U - \{u\}, V - \{v\}) \in \Gamma \cup \{(\phi, \phi)\}$, where $v = \min V$. So $\mathcal{A}_k$ is $U - \{u\}, V - \{v\}$ compressed [as every set is (ϕ,ϕ)-compressed]. Thus ✱ in lemma 6 is satisfied. Set $\mathcal{A}_{k+1} = C_{UV}(\mathcal{A}_k)$, and continue inductively. Sequence terminates, as $\sum_{A \in \mathcal{A}_k} \sum_{i \in A} 2^i$ is a strictly decreasing function of k. The set $\mathcal{B} = \mathcal{A}_k$ satisfies $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial \mathcal{B}| \le |\partial \mathcal{A}|$, and $\mathcal{B}$ is UV-compressed $\forall\ (U,V) \in \Gamma$.

Claim: $\mathcal{B} = \mathcal{C}$.

Proof: Suppose not. Then, $\exists\ A, B \in X^{(r)}$ with $A \notin \mathcal{B}$, $B \in \mathcal{B}$ and $A < B$ in colex. So $\max(A - B) < \max(B - A)$. So $C_{A-B, B-A}(B) = A$, so $A \in \mathcal{B}$. #

Remarks: (i) Equivalent form of Kruskal-Katona: if $\mathcal{A} \subset X^{(r)}$ with $|\mathcal{A}| = \binom{k_r}{r} + \binom{k_{r-1}}{r-1} + \dots + \binom{k_s}{s}$, where $s > 0$ and $k_r > k_{r-1} > \dots > k_s$, then $|\partial \mathcal{A}| \ge \binom{k_r}{r-1} + \binom{k_{r-1}}{r-2} + \dots + \binom{k_s}{s-1}$.

(ii) Proof of Theorem 7 only used lemma 6 (not lemma 4 or corollary 5).

(iii) If $|\mathcal{A}| = \binom{k}{r}$ and $|\partial \mathcal{A}| = \binom{k}{r-1}$, can check that $\mathcal{A} = Y^{(r)}$, some $Y \in X^{(k)}$. In other words, $\mathcal{A}$ is isomorphic to $\mathcal{C}$ (ie $\exists$ a permutation f of X with $\mathcal{C} = \{fA : A \in \mathcal{A}\}$). But if $|\mathcal{A}|$ is not of this form, can have (for some $|\mathcal{A}|$), $|\partial \mathcal{A}| = |\partial \mathcal{C}|$, but $\mathcal{A}$ not isomorphic to $\mathcal{C}$.

For $\mathcal{A} \subset X^{(r)}$, $0 \le r \le n-1$, the upper shadow of $\mathcal{A}$ is $\partial^+ \mathcal{A} = \{B \in X^{(r+1)} : B \supset A,\ \text{some } A \in \mathcal{A}\}$.

Note that $A < B$ in colex $\iff$ $A^c < B^c$ in lex with order on ground set reversed. Thus Theorem 7 has the following reformulation:

Corollary 8: Let $0 \le r \le n-1$ and $\mathcal{A} \subset X^{(r)}$. Let $\mathcal{C}$ be the set of the first $|\mathcal{A}|$ elements of $X^{(r)}$ in lex. Then $|\partial^+ \mathcal{A}| \ge |\partial^+ \mathcal{C}|$

Note: If $\mathcal{A}$ is an initial segment of $X^{(r)}$ in colex, then $\partial \mathcal{A}$ is an initial segment of $X^{(r-1)}$ in colex. [Eg, because if $\mathcal{A} = \{A \in X^{(r)} : A \le \{a_1, \dots, a_r\}\}$, where $a_1 < \dots < a_r$, then $\partial \mathcal{A} = \{B \in X^{(r-1)} : B \le \{a_1, \dots, a_r\}\}$]

Corollary 9: Let $1 \le r \le n$ and $\mathcal{A} \subset X^{(r)}$. Let $\mathcal{C}$ be the set of the first $|\mathcal{A}|$ elements of $X^{(r)}$ in colex. Then, $|\partial^t \mathcal{A}| \geq |\partial^t \mathcal{C}|$ $\forall\, t = 1, \dots, r$.
In particular, $|\mathcal{A}| = \binom{k}{r} \Rightarrow |\partial^t \mathcal{A}| \geq \binom{k}{r-t}$.

Proof: If $|\partial^t \mathcal{A}| \geq |\partial^t \mathcal{C}|$, then $|\partial^{t+1} \mathcal{A}| \geq |\partial^{t+1}\mathcal{C}|$, by Kruskal-Katona, (because $\partial^t \mathcal{C}$ is an initial segment of colex). So done by induction on t.

Intersecting Families.

$\mathcal{A} \subset \mathbb{P}(X)$ is *intersecting* if $A \cap B \neq \emptyset$ $\forall\, A, B \in \mathcal{A}$.
Example: $\mathcal{A} = \{A \subset X : 1 \in A\}$, $|\mathcal{A}| = 2^{n-1}$.
Is there a larger intersecting family?

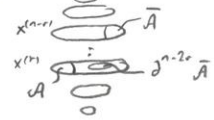
Proposition 10: Let $\mathcal{A} \subset \mathbb{P}(X)$ be intersecting. Then $|\mathcal{A}| \le 2^{n-1}$.
Proof: For any $A \subset X$, A and $X - A$ cannot both belong to $\mathcal{A}$.

Remark: The above is not the only extremal example. Eg: $\mathcal{A} = \{A \subset X : |A| > \frac{n}{2}\}$.

How large can an intersecting $\mathcal{A} \subset X^{(r)}$ be?
If $r > n/2$, can take $X^{(r)}$
If $r = n/2$, maximum size is $\frac{1}{2}\binom{n}{r}$; just do not choose A and A^c.
So look at $r < n/2$. Could try $\mathcal{A} = \{A \in X^{(r)} : 1 \in A\}$, $|\mathcal{A}| = \binom{n-1}{r-1}$.

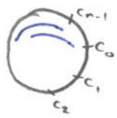
Example: In $[8]^{(3)}$: $\mathcal{A} = \{A \in [8]^{(3)} : 1 \in A\}$ has $|\mathcal{A}| = \binom{7}{2} = 21$
$\mathcal{A} = \{A \in [8]^{(3)} : |A \cap \{1,2,3\}| \geq 2\}$, has $|\mathcal{A}| = 1 + \binom{3}{2}\binom{5}{1}$ [meet $\{1,2,3\}$ in 3 or 2 places]
$= 16$.

Theorem 11 (Erdös-Ko-Rado Theorem): Let $r < n/2$ and $\mathcal{A} \subset X^{(r)}$ be intersecting. Then $|\mathcal{A}| \le \binom{n-1}{r-1}$.
Proof 1: Let $\bar{\mathcal{A}} = \{A^c : A \in \mathcal{A}\} \subset X^{(n-r)}$. Then, $\forall A \in \mathcal{A}$, $B \in \bar{\mathcal{A}}$, have $A \not\subset B$. In other words, $\mathcal{A}$ and $\partial^{n-2r}\bar{\mathcal{A}}$ are disjoint subsets of $X^{(r)}$.



Suppose $|\mathcal{A}| > \binom{n-1}{r-1}$. Then $|\bar{\mathcal{A}}| = |\mathcal{A}| > \binom{n-1}{r-1} = \binom{n-1}{n-r}$
Hence, Kruskal-Katona (Corollary 9) gives $|\partial^{n-2r}\bar{\mathcal{A}}| \geq \binom{n-1}{r}$
But, $|\mathcal{A}| + |\partial^{n-2r}\bar{\mathcal{A}}| > \binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$. ※

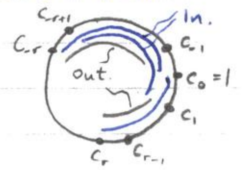
Proof 2: Consider a cyclic order, C, of X. That is, a bijection $C: X \to \mathbb{Z}_n$



Question: How many $A \in \mathcal{A}$ are intervals (set of consecutive elements) in C?
Answer: At most r, as if $\{c_1, \dots, c_r\} \in \mathcal{A}$, then, for each $1 \le i \le r-1$, at most one of $\{\dots, c_i\}$ and $\{c_{i+1}, \dots\}$ can belong to $\mathcal{A}$.
There are $n!$ cyclic orders of X. A given r-set is an interval in $nr!(n-r)!$ of these orders (choose position, then average inside and outside).
Hence $|\mathcal{A}|\, nr!\,(n-r)! \le n!\, r$, so $|\mathcal{A}| \le \frac{n!\, r}{nr!(n-r)!} = \frac{(n-1)!}{(r-1)!(n-r)!}$ $\binom{n-1}{r-1}$

Remark: Could rephrase as a count of the edges in the bipartite graph with vertex classes $\mathcal{A}$ and $\{C : C \text{ a cyclic order}\}$, with $A \in \mathcal{A}$ joined to C if A is an interval of C. This method (Proof 2) is called averaging.

Equality in Erdős-Ko-Rado: Let $\mathcal{A} \subset X^{(r)}$ be intersecting $(r < \frac{n}{2})$, $|\mathcal{A}| = \binom{n-1}{r-1}$.
Wish to show that $\mathcal{A} = \{A \in X^{(r)} : x \in A\}$, some x. For each cyclic order C, exactly r intervals of C belong to $\mathcal{A}$. So $\exists x = x(C) \in X$ belonging to all these intervals. Need to show that $x(C) = x(C')$ $\forall\, C, C'$.
Fix a cyclic order C, say $C = \cdots c_{-2}\, c_{-1}\, c_0\, c_1\, c_2 \cdots$, with wlog $x(C) = c_0 = 1$.
Have $\{c_{-k}, \ldots, c_{-k+r-1}\} \in \mathcal{A}$ for each $k = 0, \ldots, r-1$, and $\{c_{-r}, \ldots, c_{-1}\} \notin \mathcal{A}$, $\{c_1, \ldots, c_r\} \notin \mathcal{A}$. Hence, from picture, clear that if C' obtained from C by an adjacent transposition not involving 1, then $x(C') = x(C) = 1$. But, such transpositions generate (up to rotation) every cyclic order.



$\mathcal{A} \subset X^{(r)}$ is t-intersecting if $|A \cap B| \geq t$ $\forall\, A, B \in \mathcal{A}$. Only look at $t \leq r$ (else no non-empty t-intersecting families) and $2r - t < n$ (else $X^{(r)}$ is t-intersecting).

How large can $\mathcal{A}$ be?
Obvious guess: $\mathcal{A}_0 = \{A \in X^{(r)} : A \supset [t]\}$, $|\mathcal{A}_0| = \binom{n-t}{r-t}$. Also, for each $0 \leq \alpha \leq r-t$, let
$\mathcal{A}_\alpha = \{A \in X^{(r)} : |A \cap [t+2\alpha]| \geq t + \alpha\}$, clearly t-intersecting.

Examples: In $[8]^{(4)}$, $t = 2$: $\mathcal{A}_0 = \{A \in [8]^{(4)} : \{1,2\} \subset A\}$, $|\mathcal{A}_0| = \binom{6}{2} = 15$.
$\mathcal{A}_1 = \{A \in [8]^{(4)} : |A \cap \{1,2,3,4\}| \geq 3\}$, $|\mathcal{A}_1| = 1 + \binom{4}{3}\binom{4}{1} = 17$.
$\mathcal{A}_2 = [6]^{(4)}$, $|\mathcal{A}_2| = \binom{6}{4} = 15$.

In $[7]^{(4)}$, $t = 2$: $|\mathcal{A}_0| = \binom{5}{2} = 10$
$|\mathcal{A}_1| = 1 + \binom{4}{3}\binom{3}{1} = 13$
$|\mathcal{A}_2| = \binom{6}{4} = 15$.

In $[9]^{(4)}$, $t = 2$: $|\mathcal{A}_0| = \binom{7}{2} = 21$
$|\mathcal{A}_1| = 1 + \binom{4}{3}\binom{5}{1} = 21$
$|\mathcal{A}_2| = \binom{6}{4} = 15$.

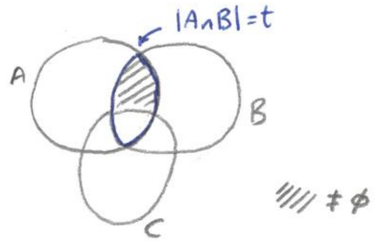
Clearly, $|\mathcal{A}_0|$ is quadratic in n, $|\mathcal{A}_1|$ is linear, $|\mathcal{A}_2|$ is constant. So expect $\mathcal{A}_0$ to win eventually.

Theorem 12: Let $1 \leq t \leq r$ and let $\mathcal{A} \subset X^{(r)}$ be t-intersecting. Then, $n \geq (16r)^r \Rightarrow |\mathcal{A}| \leq \binom{n-t}{r-t}$

Remarks: (i) $n \geq (16r)^r$ says "n sufficiently large."
(ii) The bound $\binom{n-t}{r-t}$ is best possible, eg: $\mathcal{A} = \{A \in X^{(r)} : A \supset [t]\}$.
(iii) Sometimes called the "2nd Erdős-Ko-Rado Theorem."

Idea: The extremal $\mathcal{A}$ has $r-t$ "degrees of freedom", ie, $|\mathcal{A}|$ is a polynomial in n of degree $r-t$.

<u>Proof</u>: wlog $t<r$, wlog $\mathcal{A}$ is a maximal t-intersecting family.
Then $\exists A,B \in \mathcal{A}$ with $|A \cap B| = t$ (if not then for each $A \in \mathcal{A}$, could add $A \cup \{i\} - \{j\}$, whence $A \cup \{i\} - \{j\} \in \mathcal{A}$, whence $\mathcal{A} = X^{(r)}$)



Suppose $\mathcal{A}$ is not of the form $\{A \in X^{(r)} : Y \subset A\}$, any $Y \in X^{(t)}$.
Hence $\exists C \in \mathcal{A}$ with $A \cap B \not\subset C$. So, for any $D \in \mathcal{A}$ we have $|D \cap (A \cup B \cup C)| \geq t+1$.
(Thus done, as $\leq r-(t+1)$ degrees of freedom).
<u>Crude bounds</u>: $r \leq |A \cup B \cup C| \leq 3r$.

$$\text{So, } |\mathcal{A}| \leq \underbrace{2^{3r}}_{\text{(inside)}} \Big(\underbrace{\binom{n-t}{r-t-1} + \binom{n-t}{r-t-2} + \dots + \binom{n-t}{0}}_{\text{(outside)}} \Big) \leq 2^{3r}\left((n-t)^{r-t-1} + \dots + (n-t)^0\right) \leq 2^{3r+1}(n-t)^{r-t-1}.$$

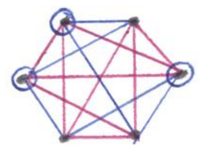
But, $\binom{n-t}{r-t} \geq \frac{(\frac{n}{2})^{r-t}}{r^r}$, so sufficient to show $\frac{n^{r-t}}{2^{r-t} \cdot r^r} \geq 2^{3r+1} \cdot (n-t)^{r-t-1}$.
So, sufficient to show $n \geq r^r \cdot 2^{4r-t+1}$, so done.

<u>Remarks</u>: (i) With more care, could obtain, eg, $n \geq 2tr^3$.
(ii) <u>Frankl conjecture</u>: Let $1 \leq t \leq r$, $2r-t<n$, and let $\mathcal{A} \subset X^{(r)}$ be t-intersecting.
Then $|\mathcal{A}| \leq \max\{|\mathcal{A}_0|, |\mathcal{A}_1|, \dots, |\mathcal{A}_{r-t}|\}$ [Recall: $\mathcal{A}_\alpha = \{A \in X^{(r)} : |A \cap [t+2\alpha]| \geq t+\alpha\}$].
First special case - <u>$4n$-$2n$ conjecture</u>: Let $\mathcal{A} \subset [4n]^{(2n)}$ be 2-intersecting. Then
$|\mathcal{A}| \leq |\{A \in [4n]^{(2n)} : |A \cap [2n]| \geq n+1\}| = \frac{1}{2}\left(\binom{4n}{2n} - \binom{2n}{n}^2\right)$.
Frankl conjecture proved by Ahlswede and Khachatrian in 1995.

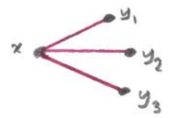
3. Ramsey Theory

<u>Slogan</u>: "Can we find some order in enough disorder?"

<u>Example</u>: Suppose we 2-colour K_6 (ie, have $C: E(K_6) \to \{1,2\}$)
Can we find a monochromatic triangle? (That is, a K_3 on which C is constant).



Choose $x \in V(K_6)$. Have $d(x)=5$, so ≥ 3 edges incident with x have the same colour. Say, wlog, xy_1, xy_2, xy_3 are red.



If some $y_i y_j$ is red, have xy_iy_j a red K_3.
If all y_iy_j are blue, have $y_1y_2y_3$ a blue K_3.

What n (if any) is such that whenever we 2-colour K_n we have a monochromatic K_4?
Write $R(s)$ for the smallest n (if it exists) such that whenever K_n is 2-coloured (ie, have $C: E(K_n) \to \{1,2\}$), $\exists$ a monochromatic K_s (ie, a K_s on which C is constant).
Equivalently, $R(s)$ is the smallest n (if it exists) such that if G is a graph on n points, then $G \supset K_s$ or $\bar{G} \supset K_s$.

<u>Aim</u>: To show $R(s)$ exists $\forall s$ (and find roughly how large it is).

<u>Example</u>: $s=3$. We know $R(s) \leq 6$. In fact, $R(s)=6$:

Idea: In proof of $R(s) \leq 6$, we used the following for $s=2$, $t=3$: for $s,t \geq 2$, write $R(s,t)$ for the smallest n (if it exists) such that whenever K_n is 2-coloured $\exists$ a red K_s or blue K_t.

Examples: (i) $R(s) = R(s,s)$

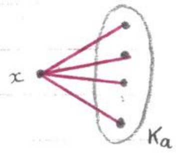
(ii) $R(s,t) = R(t,s)$

(iii) $R(s,2) = s$.

Theorem 1 (Ramsey's Theorem): $R(s,t)$ exists $\forall\, s,t$. Moreover, $R(s,t) \leq R(s-1,t) + R(s,t-1)$ $(s,t \geq 3)$

Proof: Enough to show that if $R(s-1,t)$ and $R(s,t-1)$ are finite, then $R(s,t) \leq R(s-1,t) + R(s,t-1)$. [Then, all $R(s,t)$ are finite, by induction on $s+t$]. So let $a = R(s-1,t)$, $b = R(s,t-1)$, and suppose we are given a 2-colouring of K_{a+b}. Pick $x \in V(K_{a+b})$. Then $d(x) = a+b-1$. Therefore, either $\geq a$ edges incident with x are red, or $\geq b$ are blue.

If $\geq a$ are red: Consider the K_a spanned by endpoints of the a red edges ~~…~~ incident with x. Since $a = R(s-1,t)$, this K_a contains a red K_{s-1} or a blue K_t.



If $\geq b$ are blue: similarly.

Remarks: (i) So, for any s, if n sufficiently large and G a graph on n points, then $G \supset K_s$ or $\overline{G} \supset K_s$.

(ii) Very few of the 'Ramsey numbers' $R(s,t)$ are known exactly. See later.

Corollary 2: Let $s,t \geq 2$. Then $R(s,t) \leq \binom{s+t-2}{s-1}$

Proof: Induction on $s+t$. If $s=2$ or $t=2$, then done.

For $s,t \geq 3$, $R(s,t) \leq R(s-1,t) + R(s,t-1) \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}$

What about k colours? Write $R_k(s_1,..,s_k)$ for the smallest n (if it exists) such that whenever K_n is k-coloured (ie, have $C: E(K_n) \to \{1,..,k\}$) there is a K_{s_i} with all edges coloured i, some i $(k \geq 1,\ s_1,..,s_k \geq 2)$.

Corollary 3: Let $k \geq 1$ and $s_1,..,s_k \geq 2$. Then $R_k(s_1,..,s_k)$ exists.

Proof: "Turquoise Spectacles", eg, $R(3,6) \to$ red K_3 or blue-green $K_6 \to$ blue K_3 or green K_3.

Induction on k. $k=1$, done. (also, $k=2$, done).

Given $k \geq 2$ and $s_1,..,s_k \geq 2$, let K_n be k-coloured, where $n = R(s_1, R_{k-1}(s_2,..,s_k))$. Considering this as a 2-colouring by colours "1" and "2 or 3 or .. or k", we obtain either a K_{s_1} coloured 1, or a $K_{R_{k-1}(s_2,..,s_k)}$ coloured $2,3,..,k$ (ie, $(k-1)$-coloured). Done.

Remark: Alternatively, could mirror proof of Theorem 1 to obtain

$$R_k(s_1,..,s_k) \leq R_k(s_1-1, s_2,..,s_k) + R_k(s_1, s_2-1, s_3,..,s_k) + \cdots + R_k(s_1,..,s_{k-1}, s_k-1), \quad (k \geq 2).$$

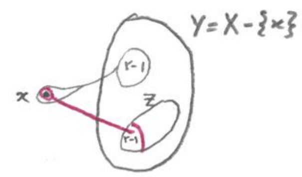
What if we colour r-sets? Write $R^{(r)}(s,t)$ for the smallest n (if it exists) such that whenever $X^{(r)}$ is 2-coloured (ie, $X = \{1,..,n\}$, $C: X^{(r)} \to \{1,2\}$), we have either a red $A \in X^{(s)}$ (ie $C(B) = 1\ \forall\, B \in A^{(r)}$) or a blue $A \in X^{(t)}$ (ie $C(B) = 2\ \forall B \in A^{(r)}$), $(r \geq 1$ and $s,t \geq r)$.

So, $R(s,t) = R^{(2)}(s,t)$. Have $R^{(r)}(s,r) = s$, $R^{(r)}(s,t) = R^{(r)}(t,s)$ and $R^{(1)}(s,t) = s+t-1$.

Theorem 4 (Ramsey for r-sets): Let $r \geq 1$ and $s,t \geq r$. Then $R^{(r)}(s,t)$ exists.

Idea: In proof for $r=2$ (Theorem 1), we used $r=1$.

Proof: Induction on r. If $r=1$, done. (Also, $r=2$, done).
Given r, we induction on $s+t$. If $s=r$ or $t=r$, done.
So, given $r \geq 2$ and $s,t \geq r$,
Claim: $R^{(r)}(s,t) \leq R^{(r-1)}\left(R^{(r)}(s-1,t), R^{(r)}(s,t-1)\right) + 1$.
Proof: Let $a = R^{(r)}(s-1,t)$, $b = R^{(r)}(s,t-1)$ and $n = R^{(r-1)}(a,b)+1$



Given a 2-colouring C of $X^{(r)}$, define a 2-colouring C' of $Y^{(r-1)}$, $(Y = X - \{x\})$, by $C'(A) = C(A \cup \{x\})$. So, by definition of $R^{(r-1)}(a,b)$, we have either a red a-set for C' or a blue b-set for C'. By symmetry, wlog, we have a red a-set Z for C'. Thus $\forall A \in Z^{(r-1)}$, we have $C(A \cup \{x\}) =$ red.
By definition of $R^{(r)}(s-1,t)$, Z contains either an $(s-1)$-set, all of whose r-sets are red, or a t-set, all of whose r-sets are blue. (for the colouring C).
If a blue t-set: done
If a red $(s-1)$-set: add x to obtain a red s-set. Done.

Remarks: (i) Similar result for k-colourings of $X^{(r)}$, eg by "turquoise ~~scattering~~ spectacles"
(ii) The bounds we obtain for $R^{(r)}(s,t)$ are rather large:
Define $f_1, f_2, \ldots : \mathbb{N} \to \mathbb{N}$ by: $f_1(x) = 2x$, $x \in \mathbb{N}$, and for $n \geq 2$, $f_n(x) = \underbrace{f_{n-1}(f_{n-1}(\cdots(f_{n-1}(1))\cdots))}_{x}$.
So $f_2(x) = 2^x$. $f_3(x) = 2^{2^{\cdot^{\cdot^{2}}}}$, a tower of height x. f_4? $f_4(1) = 2$, $f_4(2) = 2^2 = 4$, $f_4(3) = 2^{2^{2^2}} = 2^{16}, \ldots$
Our bound for $R^{(r)}(s,t)$ from the proof of Theorem 4 is of the form $f_{k-1}(s+t)$.
(Much more, really)

Infinite Ramsey Theory

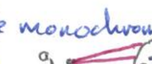
Let $\mathbb{N}^{(2)}$ be 2-coloured, (ie, have $C: \mathbb{N}^{(2)} \to \{1,2\}$). Must there exist an infinite $M \subset \mathbb{N}$ that is monochromatic (ie, c constant on M).

Examples: (i) Colour ij red if $i+j$ even, blue if $i+j$ odd. Take $M = \{n \in \mathbb{N} : n \text{ is even}\}$.
(ii) Colour ij red if $\max\{n : 2^n \mid i+j\}$ is even, blue if it is odd. Take $M = \{2^n : n \text{ even}\}$.
(iii) Colour ij red if $i+j$ has an even number of (distinct) prime factors, and blue if odd.

We know that some colour has arbitrarily large finite monochromatic sets, but this does not imply an infinite set of that colour. **Eg:**  ... etc.

Theorem 5 (Ramsey's Theorem, Infinite Version): Let $\mathbb{N}^{(2)}$ be 2-coloured. Then $\exists$ infinite monochromatic $M \subset \mathbb{N}$.

Proof: Choose $a_1 \in \mathbb{N}$. Then $\exists$ an infinite $B_1 \subset \mathbb{N} - \{a_1\}$ such that all edges a_1b: $b \in B_1$



have the same colour, say c_1. Choose $a_2 \in B_1$. Then $\exists$ an infinite $B_2 \subset B_1 - \{a_2\}$ such that all edges a_2b: $b \in B_2$ have the same colour, say c_2. Continue inductively. We obtain a sequence $a_1, a_2, \ldots$ in $\mathbb{N}$ and a sequence $c_1, c_2, \ldots$ of colours such that a_ia_j $(i<j)$ has colour c_i.

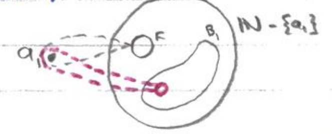
Some colour occurs infinitely often as a c_i, say $c_{i_1} = c_{i_2} = \ldots$ Then $\{a_{i_1}, a_{i_2}, \ldots\}$ is monochromatic.

Remarks: (i) Similarly for k colours. Either by same proof, or by "turquoise spectacles".
(ii) In example (iii) above, no example is known.

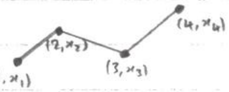
Example: Any sequence $x_1, x_2, \dots$ in a totally ordered set has a monotone subsequence.
Proof: Colour $\mathbb{N}^{(2)}$ by giving ij $(i<j)$ colour up if $x_i < x_j$ and down if $x_i \geq x_j$.
Theorem 5 gives an infinite monochromatic set.

What about $\mathbb{N}^{(r)}$? Eg: $r=3$. Colour i,j,k $(i<j<k)$ red if $i \mid j+k$, blue otherwise. Take $M = \{2^n : n \in \mathbb{N}\}$.

Theorem 6 (Infinite Ramsey for r-sets): Let $\mathbb{N}^{(r)}$ be 2-coloured (ie, have $c: \mathbb{N}^{(r)} \to \{1,2\}$). Then $\exists$ $M \subset \mathbb{N}$ such that M monochromatic (ie, c constant on $M^{(r)}$).



Proof: Induction on r. $r=1$, done. (And, $r=2$, done).
Choose $a_1 \in \mathbb{N}$. This induces a 2-colouring of $(\mathbb{N} - \{a_1\})^{(r-1)}$,
by $c'(F) = c(F \cup \{a_1\})$, $F \in (\mathbb{N} - \{a_1\})^{(r-1)}$. By induction hypothesis,
$\exists$ infinite $B_1 \subset \mathbb{N} - \{a_1\}$, monochromatic for c'. Ie, all sets of the form $F \cup \{a_1\}$, $F \in B_1^{(r-1)}$, have the same colour, say c_1.
Choose $a_2 \in B_1$. We obtain an infinite $B_2 \subset B_1 - \{a_2\}$ such that all sets of the form $F \cup \{a_2\}$, $F \in B_2^{(r-1)}$ have the same colour, say c_2. Continue inductively.
We obtain $a_1, a_2, \dots$ in $\mathbb{N}$ and colours $c_1, c_2, \dots$ such that for any r-set $a_{i_1} \dots a_{i_r}$ $(i_1 < \dots < i_r)$, we have $c(a_{i_1}, \dots, a_{i_r}) = c_{i_1}$. But, infinitely many of the c_i agree: $c_{i_1} = c_{i_2} = \dots$
Then $\{a_{i_1}, a_{i_2}, \dots\}$ is monochromatic.



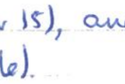
Example: We know that, given $(1, x_1), (2, x_2), (3, x_3), \dots$ in $\mathbb{R}^2$, $\exists$ subsequence such that the 'induced function' (ie, piecewise linear) is monotone.
Can guarantee that the induced function is ~~monotone~~ convex or concave: colour $\mathbb{N}^{(3)}$ by giving ijk $(i<j<k)$ the colour convex if we have x_i x_j x_k, and colour concave if we have x_i x_j x_k, and apply Theorem 6 to obtain an infinite monochromatic subset.

Bounds on Ramsey Numbers.

Very few of the $R(s,t)$ are known $(s,t \geq 3)$: $R(3,3)=6$, $R(3,4)=9$, $R(3,5)=14$, $R(3,6)=18$, $R(3,7)=23$, $R(4,4)=18$, $R(4,5)=25$.

Example: $R(4,4) > 17$: $G = \mathbb{Z}_{17}$, colour xy red is $x-y$ is a quadratic residue mod 17, blue if not. (Must check $\nexists$ monochromatic K_4).

For $k \geq 3$ colours, only one non-trivial Ramsey number is known: $R_3(3,3,3) = 17$.



$R_3(3,3,3) \leq 17$: Choose x; $d(x) = 16$, so ≥ 6 edges from x with same colour, say red:
If any of the K_6 red, done. If not, have a blue-green K_6, so done.
$R_3(3,3,3) > 16$: Let F be the field of order 16, and colour xy by the cubic character of $x-y$.
In other words, choose a generator g for F^* (a cyclic group of order 15), and give $x-y = g^i$ the colour i mod 3. (Must check $\nexists$ monochromatic triangle).

For colourings of $[n]^{(r)}$, no numbers are known. It is known that $13 \le R^{(3)}(4,4) \le 15$.
No $R(s,s)$ is known for $s \geq 5$. Known that $43 \le R(5,5) \le 49$. Check on computer?
To see if $R(5,5) > 43$, we would have to check $\binom{43}{5}$ 5-sets in $2^{\binom{43}{2}}$ graphs.

Upper bound on $R(s,s)$.

We know $R(s,s) \leqslant \binom{2s-2}{s-1}$ by corollary 2. So $R(s,s) < 2^{2s}$ (as $\binom{n}{k} \le 2^n$).
Stirling would give $R(s,s) < \frac{2^{2s}}{\sqrt{s}} \cdot \frac{1}{2\sqrt{\pi}}$, $s \geqslant 4$.

Lower bound on $R(s,s)$.

Theorem 7: Let $s \geqslant 3$. Then $R(s,s) > 2^{s/2}$.

Proof: Given n, choose a red-blue colouring of K_n at random, ie, each edge red with probability $\frac{1}{2}$, independently. Have $\binom{n}{s}$ s-sets. For a fixed s-set Y, $P(Y \text{ monochromatic}) = \frac{2}{2^{\binom{s}{2}}}$. So $P(\exists \text{ monochromatic s-set}) \le \binom{n}{s} 2^{1-\binom{s}{2}}$.
Then, certainly, $R(s,s) > n$ if $\binom{n}{s} \cdot 2^{1-\binom{s}{2}} < 1$.
Have $\binom{n}{s} < \frac{n^s}{s!}$. So done if $\frac{n^s}{s!} < 2^{\frac{s^2-s}{2}-1}$. But $s! \geqslant 2^{\frac{s}{2}+1}$. So done if $n^s \le 2^{s^2/2}$.

Remarks: (i) Equivalently, have $2^{\binom{n}{2}}$ colourings, and each s-set monochromatic in $2^{\binom{n}{2}-\binom{s}{2}}$, so done if $2^{\binom{n}{2}-\binom{s}{2}+1} \cdot \binom{n}{s} < 2^{\binom{n}{2}}$
(ii) The above proof is a "Random Graphs" proof
(iii) Proof gives no clue as to how to construct such a colouring.
(iv) No construction is known giving an exponential lower bound on $R(s,s)$.

So, $\sqrt{2}^s < R(s,s) < 4^s$, ie $R(s,s)$ grows exponentially.
Best known bounds are $\sqrt{2}^s < R(s,s) < \frac{4^s}{s}$ (upper bound due to Thomson).
Why is nothing better known?
$R(s,s) > n$ says $\exists$ a 'bad' (no monochromatic K_s) colouring of K_n, ie a very disorderly colouring. To construct such a colouring, could use induction - but this always gives some 'orderly' parts to the graph. Or could write it down directly - but how to avoid order? Try algebra, eg quadratic residues, to give a disorderly (hopefully) graph. The 'worst' graphs would seem to be random graphs - disorderly, with high probability, by definition. So, to find a graph G on 10^6 points with no K^{40} in G or $\bar{G}$, best to pick one at random.
$R(5,5)$ is far from being known. It has been "conjectured" that $R(6,6)$ will never be known.