ALGEBRAIC TOPOLOGY EXAMPLE SHEET 1 W.B.R.L. 1995

1. Let $a : S^n \longrightarrow S^n$ be the antipodal map (defined by $a(x) = -x$). Prove that a is homotopic to the identity if n is odd.
[Hint. Consider $n = 1$ first. Later in the course it will be shown that a is not homotopic to the identity if n is even.]

2. Let X be a contractible space and let Y be any space. Show that
(i) X is path connected;
(ii) $X \times Y$ is homotopy equivalent to Y;
(iii) any two maps from Y to X are homotopic;
(iv) if Y is path connected, any two maps from X to Y are homotopic.

3*. Let X be the subset of $\mathbb{R}^2$ that consists of all points on all line segments from the point $(0,1)$ to a point of the form $(x,0)$ where x is rational and $0 \leq x \leq 1$. Show that X is contractible. Show also that the point $(0,0)$ must 'move' in any homotopy between the identity map on X and the constant map sending all of X to $(0,0)$.
Find a contractible subspace Y of $\mathbb{R}^2$ with the property that every point of Y has to move in any contracting homotopy.

4. Show that the torus less one point, the Klein bottle less one point and $\mathbb{R}^2$ less two points are each homotopy equivalent to $S^1 \vee S^1$ (the space obtained by gluing two disjoint circles together at a point).
[Hint. Draw pictures showing how $S^1 \vee S^1$ can be embedded as a retract in each of the other spaces and describe homotopies with words rather than formulae.]

5. Let (X, x_0) and (Y, y_0) spaces equipped with base points. Show that

$$\Pi_1(X \times Y, (x_0, y_0)) \cong \Pi_1(X, x_0) \times \Pi_1(Y, y_0) .$$

6. Let G be a space equipped with a continuous multiplication $m : G \times G \to G$ and a point $e \in G$ which acts as an identity (that is, $m(e,g) = g = m(g,e)$ for all $g \in G$). [The most familiar examples are topological groups, but associativity and inverses play no part in what follows.] For any pair of loops u and v based at the e, let $u \star v$ be the loop defined by $(u \star v)(s) = m(u(s), v(s))$ for all $s \in I$. Prove that $u \cdot v$, $u \star v$, and $v \cdot u$ are all homotopic relative to $\{0,1\}$ and deduce that $\Pi_1(G,e)$ is abelian.

7. Show that a path connected space X is simply connected if and only if every continuous map $f : S^1 \to X$ can be extended over B^2 .

8. Regarding S^1 as the unit complex numbers, describe the homomorphisms $f_\star : \Pi_1(S^1, 1) \longrightarrow \Pi_1(S^1, f(1))$ when

(i) $f(e^{i\theta}) = e^{i(\theta+\pi/2)}$;

(ii) $f(e^{i\theta}) = e^{in\theta}$ for a fixed integer n;

(iii) $f(e^{i\theta}) = \begin{cases} e^{i\theta}, & \text{if } 0 \le \theta \le \pi \text{ ;} \\ e^{i(2\pi-\theta)}, & \text{if } \pi \le \theta \le 2\pi \text{ .} \end{cases}$

9. Consider non-trivial polynomials with complex coefficients

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0 \qquad \text{and} \qquad q(z) = z^n \ .$$

Let $C_r = \{z : |z| = r\}$. Show that, for sufficiently large values of r, $p|C_r : C_r \to \mathbb{C} - \{0\}$ and $q|C_r : C_r \to \mathbb{C} - \{0\}$ are homotopic. Deduce the Fundamental Theorem of Algebra, namely that $p(z) = 0$ for some $z \in \mathbb{C}$.

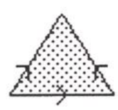
10. Let $f : (X, x_0) \to (X, x_0)$ be a map that is homotopic to the identity. Show that $f_\star : \Pi_1(X, x_0) \to \Pi_1(X, x_0)$ is an inner automorphism.

11. Let N and S be the poles of the n-sphere S^n. Show that any path is a composite of paths each lying in either $S^n - N$ or $S^n - S$; if $n \ge 2$ deduce that $\Pi_1(S^n, N)$ is trivial.

12. Prove that no two of the spaces $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$ are homeomorphic. [Hint: Consider $\mathbb{R}^n$ less one point.]

13. Prove that if f and $g : S^n \to X$ are homotopic maps then $X \cup_f B^{n+1}$ and $X \cup_g B^{n+1}$ are homotopy equivalent (here $X \cup_f B^{n+1}$ denotes the space formed from the disjoint union of X and B^{n+1} by identifying x and $f(x)$ for each $x \in S^n$).

14. The 'topologist's dunce cap' is defined to be the space D formed by identifying together the sides of a triangle respecting the directions in the way shown in the diagram.



Show that D is contractible. [Hint. Use the last question.]

Appropriate Tripos Questions: 89109, 91109, 93209.

ALGEBRAIC TOPOLOGY EXAMPLE SHEET 2 W.B.R.L. 1995

1. Construct a covering map from $\mathbb{R}^2$ to the Klein bottle K and use it to show that $\Pi_1(K)$ is isomorphic to the group whose elements are pairs (m,n) of integers with the non-abelian group operation given by

$$(m,n)\star(p,q) = (m+(-1)^n p, \quad n+q)\ .$$

2. Let G be a finite subgroup of the group $O(n)$ of $n\times n$ orthogonal matrices and suppose that no element of G, other than the identity, has 1 as an eigenvalue. Let S^{n-1}/G be the quotient of S^{n-1} by the equivalence relation given by $x\sim y$ if and only if $x = Ay$ for some $A\in G$. Show that the quotient map $S^{n-1}\to S^{n-1}/G$ is a covering map and deduce that, if $n\geq 3$, $\Pi_1(S^{n-1}/G)\cong G$. Deduce that there is a quotient of S^3 whose fundamental group is a non-abelian group of order eight [quaternions might help].

3*. Let G be the free (non-abelian) group on two generators a and b, (thus the elements of G are all formal finite products $a^{m_1}b^{n_1}a^{m_2}b^{n_2}\dots a^{m_k}b^{n_k}$ where the m_i and n_i are integers). Consider the (infinite!) 1-dimensional abstract simplicial complex K whose vertices are the elements of G, with $\{x,y\}$ being the vertices of a 1-simplex if and only if xy^{-1} is one of a, b, a^{-1} or b^{-1}. (Thus there are four 1-simplices meeting at each vertex x, their other ends being at ax,bx,$a^{-1}x$ and $b^{-1}x$.) Show that $|K|$ is contractible [hint: for each point x of $|K|$, there is a unique path from x to the vertex corresponding to the identity element of G that does not involve 'backtracking'], and that there is a covering map $|K|\to(S^1\vee S^1)$. Deduce that $\Pi_1(S^1\vee S^1)\cong G$.

4*. Let G be the free group on two generators a and b as considered in the previous question, G being (isomorphic to) the fundamental group of $S^1\vee S^1$. Draw a diagram of the covering space of $S^1\vee S^1$ corresponding to the subgroup H of G in each of the following cases.

(i) H is the subgroup consisting of powers of a only.
(ii) H is the smallest normal subgroup containing a.
(iii) H is the identity element.
(iv) H is all products of an even total number of a's, b's and their inverses.
(v) H is the commutator subgroup of G.

5. Use the Simplicial Approximation Theorem to show:

(i) if X and Y are polyhedra then there are only countably many homotopy classes of continuous maps $X\to Y$;

(ii) if $m < n$ then any continuous map $S^m \to S^n$ is homotopic to a constant map.

6. Show that the fundamental group of a polyhedron depends only on its 2-skeleton: that is, for any simplicial complex K and vertex a of K, we have $\Pi_1(|K|, a) \cong \Pi_1(|K_{(2)}|, a)$ where $K_{(2)}$ is the 2-skeleton of K. [Hint. Apply the Simplicial Approximation Theorem to paths in $|K|$ and homotopies between them]

7. Let K and L be simplicial complexes. Prove that $|K| \times |L|$ is a polyhedron. [Method: construct a triangulation whose vertices are pairs $(\widehat{\sigma}, \widehat{\tau})$ where σ is a simplex of K and τ a simplex of L, and whose simplices are spanned by sequences of vertices $((\widehat{\sigma}_0, \widehat{\tau}_0), \ldots, (\widehat{\sigma}_n, \widehat{\tau}_n))$ such that for each i we have $\sigma_{i-1} \leq \sigma_i$ and $\tau_{i-1} \leq \tau_i$, at least one of these two inequalities being proper. You may find it helpful to consider first what this gives when both $|K|$ and $|L|$ are 1-simplices.]

8. Show that it is possible to choose an infinite sequence of points $\{x_1, x_2, x_3, ...\}$ in $\mathbb{R}^m$ which are in general position in the sense that no $m+1$ of them lie in a proper affine subspace (i.e. a coset of a proper vector subspace). Deduce that, if K is an abstract (finite) simplicial complex having no simplices of dimension greater than n, it is possible to find a (geometric) simplicial complex in $\mathbb{R}^{2n+1}$ that is isomorphic to K . [This result is best possible: it can be shown that the n-skeleton of a $(2n+2)$-simplex cannot be realized in $\mathbb{R}^{2n}$.]

9. Let K be a simplicial complex satisfying the following conditions:

(i) K has no simplices of dimension greater than n;

(ii) every simplex of K is a face of some n-simplex;

(iii) every $(n-1)$-simplex of K is a face of exactly two n-simplices;

(iv) for any two n-simplices σ and τ of K, there exists a finite sequence of n-simplices, beginning with σ and ending with τ, in which each adjacent pair of simplices have a common $(n-1)$-dimensional face.

Show that $H_n(K)$ is either $\mathbb{Z}$ or the trivial group, and that in the former case it is generated by a cycle which is the sum of all the n-simplices of K, with suitable orientations.

Appropriate Tripos Questions: 88309, 89209, 90109 (omit (c)), 90209.

ALGEBRAIC TOPOLOGY EXAMPLE SHEET 3 W.B.R.L. 1995

1. For each of the following exact sequences of abelian groups and homomorphisms (in which C_r denotes the cyclic group of order r), say as much as you can about the unknown group G and/or the unknown homomorphism α:

(i) $0 \to C_2 \to G \to \mathbb{Z} \to 0$;

(ii) $0 \to \mathbb{Z} \to G \to C_2 \to 0$;

(iii) $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus C_2 \to 0$;

(iv) $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to C_2 \to 0$;

(v) $0 \to C_3 \to G \to C_2 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$.

2. Use the Mayer-Vietoris theorem to calculate the homology groups of the following spaces. [You may assume that suitable triangulations exist in each case.]

(i) The Klein bottle K, regarded as the space obtained by glueing together two copies of $S^1 \times I$.

(ii) The space X obtained by removing the interior of a small disc from a torus.

(iii) The space Y obtained from the space X of part (ii) and a Mobius band M by identifying the boundary of M with the edge of the 'hole' in X.

(iv) The space $L_n = S^1 \cup_{f_n} B^2$, where $f_n : S^1 \to S^1$ is the map $z \mapsto z^n$.

3. By restricting the (evident) homeomorphism $B^{r+s+2} \cong B^{r+1} \times B^{s+1}$ to the boundaries of these two spaces, and assuming the existence of suitable triangulations, show that we can triangulate S^{r+s+1} as the union of two subcomplexes L and M, where $|L| \simeq S^r$, $|M| \simeq S^s$ and $|L \cap M| \cong S^r \times S^s$. Use this to calculate the homology groups of $S^r \times S^s$ for $r, s \geq 1$. [Distinguish between the cases $r = s$ and $r \neq s$.]

4. Let A be a 2×2 matrix with integer entries. Show that the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ represented by A respects the equivalence relation ' $\sim$ ' on $\mathbb{R}^2$ given by $(x, y) \sim (z, t)$ if and only if $x - y$ and $z - t$ are integers, and deduce that it induces a continuous map $f_A : T^2 \to T^2$. Calculate the effect of f_A on the homology groups of T^2.

5. Suppose that a complex K is the union of subcomplexes L and M and that a complex P is the union of subcomplexes Q and R. Let $f : |K| \to |P|$ be a map such that $|Q| \supset f|L|$ and $|R| \supset f|M|$. Show that there is a commutative diagram

$$\begin{array}{ccccccccc} \ldots \to & H_r(L \cap M) & \to & H_r(L) \oplus H_r(M) & \to & H_r(K) & \to & H_{r-1}(L \cap M) & \to \ldots \\ & f_\star \downarrow & & f_\star \oplus f_\star \downarrow & & f_\star \downarrow & & f_\star \downarrow & \\ \ldots \to & H_r(Q \cap R) & \to & H_r(Q) \oplus H_r(R) & \to & H_r(P) & \to & H_{r-1}(Q \cap R) & \to \ldots \end{array}$$

in which the rows are the relevant Mayer-Vietoris sequences.

6. Let K be a simplicial complex in $\mathbb{R}^m$. The *suspension* SK of K is the complex in $\mathbb{R}^{m+1}$ whose vertices are those of K (regarded as lying in $\mathbb{R}^m \times \{0\}$) and the two points $(0, \ldots, 0, \pm 1)$, and whose simplices are those of K together with those spanned by the vertices of a simplex of K plus one or other (but not both) of the two new vertices.

(i) Verify that SK is a simplicial complex, and show in particular that if $|K| \cong S^n$ then $|SK| \cong S^{n+1}$.

(ii) Use the Mayer-Vietoris theorem to show that $H_r(SK) \cong H_{r-1}(K)$ for $r \geq 2$, and that $H_1(SK) = 0$ if K is connected.

(iii) Let $f : |K| \to |K|$ be a simplicial map, and let $\tilde{f} : |SK| \to |SK|$ be the unique extension of f to a simplicial map which interchanges the two vertices $(0, \ldots, 0, \pm 1)$. Show that, if we identify $H_r(SK)$ with $H_{r-1}(K)$, then $\tilde{f}_\star : H_r(SK) \to H_r(SK)$ sends a homology class c t $-f_\star(c)$. Deduce that if $a : S^n \to S^n$ is the antipodal map, then $a_\star : H_n(S^n) \to H_n(S^n)$ is multiplication by $(-1)^{n+1}$. [Compare question 1 on sheet 1.]

7. By considering S^n as the union of the subsets given by the inequalities $|x_{n+1}| \leq \frac{1}{2}$ and $|x_{n+1}| \geq \frac{1}{2}$, and by using th results of previous questions, show that the homology groups of real projective space $\mathbb{R}P^n$ are given by

$$H_r(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} , & \text{if } r = 0, \text{ or if } r = n \text{ and } n \text{ is odd}, \\ 0 , & \text{if } r > n \text{ or } r \text{ is even and } r \neq 0 , \\ C_2 , & \text{if } r \text{ is odd and } 0 < r < n . \end{cases}$$

[You may assume the existence of suitable triangulations.]

8. Let $G_1, G_2, \ldots, G_n$ be any finite sequence of finitely-generated abelian groups. Show that there is a connected simplicial complex K, of dimension at most $n+1$, with $H_i(K) \cong G_i$ fr $1 \leq i \leq n$ and $H_i(K) = 0$ for $i > n$. [Use questions 2(iv) and 6(ii); you may assume the result that any finitely-generated abelian group may be expressed as a direct sum of (finite or infinite) cyclic groups.]

9. Let M be an orientable closed combinatorial surface. A homeomorphism $f : M \to M$ is said to be *orientation-preserving* if $f_\star : H_2(M) \to H_2(M)$ is the identity map. Show that the orientation-preserving homeomorphisms form a subgroup of index 2 in the group of all homeomorphisms $M \to M$. Show further that there is an orientation-reversing homeomorphism $f : M \to M$ whose square is the identity, such that the quotient of M by the equivalence relation which identifies each $x \in M$ with $f(x)$ is a non-orientable closed surface . Which non-orientable surfaces can arise in this way?

Appropriate Tripos Questions: 88109, 90309, 91209, 92309, 93109.

wbrl@pmms.cam.ac.uk