Notes on Topological Spaces

All the spaces we meet in the Algebraic Topology course will be metrizable; in fact almost all of them will be subspaces of (finite-dimensional) Euclidean spaces. However, it is generally convenient to think of them as topological spaces rather than metric spaces. Since the new Tripos does not provide students with a formal introduction to topological spaces before this stage, it seems sensible to provide a brief résumé of the basic facts about them which we shall assume.

Recall that, if (X, d) is a metric space, a subset U of X is said to be *open* if, for every $x \in U$, there exists $\epsilon > 0$ such that the open ball $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$ is contained in U. It is easy to check the following properties:

(i) The empty set $\emptyset$ and the whole space X are open.

(ii) An arbitrary union of open sets is open.

(iii) A finite intersection of open sets is open.

Formally, a *topology* on a set X is a collection of subsets of X, called *open sets*, satisfying properties (i–iii) above. A *topological space* is a set equipped with a topology. We say a topological space X is *metrizable* if there is a metric d on X for which the open sets are exactly the given ones. Not every topological space is metrizable: a good example is the *indiscrete topology* in which the only open sets are $\emptyset$ and X, which cannot be metrizable if X has more than one element [exercise: why not?]. On the other hand, many different metrics can give rise to the same topology: for example, on the Euclidean space $\mathbf{R}^n$ the three metrics d_1, d_2 and d_∞ given by

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i| \ ,$$

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \ , \text{ and}$$

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{i=1}^{n} |x_i - y_i|$$

all give rise to the same notion of open set. We say that two metrics are *equivalent* if they give rise to the same topology; as far as we are concerned in this course, we do not really want to distinguish between metric spaces (X, d) and (X, d') if d and d' are equivalent.

As you know, continuity of functions between metric spaces can be defined in terms of open sets: $f \colon X \to Y$ is continuous iff $f^{-1}(V)$ is open in X for every open $V \subseteq Y$. We adopt the same definition of continuity for mappings between topological spaces. Similarly, many important properties of metric spaces such as compactness and connectedness are defined in terms of properties of their open subsets; and so these properties can be extended without difficulty to topological spaces. [However, you should *not* assume that all the theorems about compactness and connectedness in metric spaces, which you encountered in the IB

Analysis course, remain true for topological spaces.] All such properties are invariant under homeomorphism; recall that a *homeomorphism* $X \to Y$ is a bijective mapping which is continuous in both directions. We write $X \cong Y$ to denote that the spaces X and Y are homeomorphic. [Remember that a bijective mapping may be continuous in one direction but not in the other; an example is the mapping $t \mapsto e^{2\pi it}$ from the half-open interval $[0,1) \subseteq \mathbf{R}$ to the circle $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$.]

We shall say that a topological space X is *subeuclidean* (this is not a standard term!) if it is homeomorphic to a subset of $\mathbf{R}^n$ (for some n) with (the topology induced by) the restriction of one of the three metrics d_1, d_2 or d_∞ mentioned above. Not every metrizable space is subeuclidean, although it's difficult to give counterexamples at this stage apart from 'trivial' ones such as the set $P(\mathbf{R})$ of all subsets of $\mathbf{R}$ with the discrete metric ($d(A, B) = 1$ whenever $A \neq B$); this can't be subeuclidean because $P(\mathbf{R})$ doesn't map injectively into $\mathbf{R}^n$ for any n. As mentioned earlier, almost all the spaces we shall meet in this course will be subeuclidean, although they will not always be given explicitly in the form of subspaces of $\mathbf{R}^n$. Therefore, it will be useful to develop a few techniques for recognizing that spaces are subeuclidean.

If X and Y are topological spaces, the cartesian product $X \times Y$ is made into a topological space by declaring its open subsets to be all those subsets expressible as unions of 'open rectangles' $U \times V$, where U is open in X and V is open in Y. (Note that a finite intersection of open rectangles is an open rectangle, which is needed to verify condition (iii). Note also that this topology makes the projections $X \times Y \to X$ and $X \times Y \to Y$ continuous; indeed, it is the smallest topology on $X \times Y$, in the sense of having fewest open sets, which does this.) If the topologies on X and Y are induced by metrics d and d' respectively, then that on $X \times Y$ may be induced by any of the three equivalent metrics

$$d_1((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d'(y_1, y_2) ,$$

$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(d(x_1, x_2)^2 + d'(y_1, y_2)^2)} , \text{ or}$$

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d'(y_1, y_2)\} .$$

Thus *a product of metrizable spaces is metrizable*. It is also clear that *a product of subeuclidean spaces is subeuclidean*: if X and Y are homeomorphic to subspaces of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, then $X \times Y$ is homeomorphic to a subspace of $\mathbf{R}^{m+n}$.

If Y is a subset of a topological space X, we define the *subspace topology* on Y to consist of all sets of the form $U \cap Y$ where U is open in X; it is trivial to verify that this is a topology on Y, and that it makes the inclusion $Y \to X$ continuous (once again, it is the smallest topology on Y which does so). Once again, *a subspace of a metrizable space is metrizable*: if the topology on X is induced by a metric d, then that on Y is induced by the restriction of d to $Y \times Y \subseteq X \times X$. It is even easier to see that *a subspace of a subeuclidean space is subeuclidean.*

If X and Y are *disjoint* topological spaces, we can make their union $X \cup Y$ into a topological space by taking its open subsets to be all sets of the form $U \cup V$ where U is open in X and V is open in Y. Again, it's clear that this is a topology; in fact the unique topology which makes both X and Y into open subspaces of $X \cup Y$. [Recall also that if X

and Y are not disjoint, we can always 'make them disjoint' by replacing them by the sets $X \times \{0\}$ and $Y \times \{1\}$, on which we can impose topologies homeomorphic to the original ones.] To show that *a disjoint union of metrizable spaces is metrizable*, recall first that every metric is equivalent to a bounded metric, i.e. one satisfying $d(x, y) \leq k$ for all x, y and some fixed k—in fact we can take $k = 1$: we simply have to replace our original metric (e, say) by

$$d(x, y) = \min\{e(x, y), 1\} .$$

Now, given metrics d, d' on X, Y respectively which are both bounded by 1, we can define d'' on $X \cup Y$ by $d''(x, y) = d(x, y)$ if $x, y \in X$, $= d'(x, y)$ if $x, y \in Y$, $= 1$ otherwise. It is easy to check that this induces the correct topology on $X \cup Y$. Similarly, *a disjoint union of subeuclidean spaces is subeuclidean*; if X and Y are homeomorphic to subspaces $\overline{X}$ and $\overline{Y}$ of $\mathbf{R}^n$ (we may assume they are in Euclidean space of the same dimension, since $\mathbf{R}^m$ is homeomorphic to the subspace $\mathbf{R}^m \times \{0\}$ of $\mathbf{R}^{m+1}$), then $X \cup Y$ is homeomorphic to the subspace $\overline{X} \times \{0\} \cup \overline{Y} \times \{1\}$ of $\mathbf{R}^{n+1}$.

If X is a topological space and R is an equivalence relation on X, we define the *quotient topology* on X/R (the quotient of X by R, i.e. the set of R-equivalence classes) by declaring $V \subseteq X/R$ to be open iff $\bigcup\{C \mid C \in V\} = \{x \in X \mid [x] \in V\}$ is open in X. (Here $[x]$ denotes the R-equivalence class of x.) It is easy to verify that this is a topology on X/R, and that it makes the quotient map $x \mapsto [x]$ continuous—indeed, it is the unique *largest* topology for which this map is continuous. Unlike products, subspaces and disjoint unions, the operation of passing to a quotient space can take us out of the realm of metrizable spaces. Let $X = \mathbf{R}$, and let R be the equivalence relation of congruence mod $\mathbf{Q}$, i.e. xRy iff $x - y$ is rational. Then the quotient topology on X/R is indiscrete (because every nonempty open set in $\mathbf{R}$ meets each coset of the additive subgroup $\mathbf{Q}$), but it has more than one point (indeed, it is uncountable, since each R-equivalence class is countable), and so it is not metrizable.

Nevertheless, there are important particular cases of the quotient construction which preserve metrizability, and we shall make a good deal of use of them in the course. One such is the construction of 'glueing two spaces together along a map': suppose given spaces X and Y and a continuous map $f: Z \to X$, where Z is a subspace of Y. The space $X \cup_f Y$ obtained by *glueing Y to X along f* is the quotient of the disjoint union of X and Y by the smallest equivalence relation which identifies z with $f(z)$ for each $z \in Z$. Note that, although this relation may identify distinct points of Z if they have the same image under f, it cannot identify distinct points of X; thus the points of $X \cup_f Y$ correspond bijectively to those of the disjoint union of X and $Y \setminus Z$. It can be shown that *if X and Y are metrizable and the domain Z of f is compact, then $X \cup_f Y$ is metrizable*: we shall not prove this here, since the proof is quite involved and there is no reason why you should know it, but you are hereby warned that we shall be meeting lots of examples of spaces that are constructed in this way. The above result shows that there is no need to be frightened of them.

Example 1. n-dimensional (real) *projective space* P^n is defined as the quotient of $\mathbf{R}^{n+1} \setminus \{0\}$ by the equivalence relation which identifies two points iff they lie on the same straight line through the origin. However, since every point $\mathbf{x}$ is identified with the point $\mathbf{x}/\|\mathbf{x}\|$, we may equivalently regard it as the quotient of the unit sphere $S^n = \{\mathbf{x} \in \mathbf{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$ by the equivalence relation which identifies $\mathbf{x}$ with $-\mathbf{x}$ for each $\mathbf{x}$. (It is not obvious that the topology which this set acquires as a quotient of the subeuclidean space S^n is the same as that which P^n acquires as a quotient of $\mathbf{R}^{n+1} \setminus \{\mathbf{0}\}$, but it happens to be true.) We may further restrict our attention to points in the (closed) upper hemisphere $\{\mathbf{x} \in S^n \mid x_{n+1} \geq 0\}$ of S^n, which is homeomorphic to the unit ball $B^n = \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x}\| \leq 1\}$; and the only identifications which have to be made to points in this hemisphere is the identification of opposite points on the 'equator' $\{\mathbf{x} \in S^n \mid x_{n+1} = 0\}$, which is a copy of S^{n-1}. Thus we deduce that P^n is homeomorphic to the space $P^{n-1} \cup_q B^n$, where $q \colon S^{n-1} \to P^{n-1}$ is the quotient map. Since S^{n-1} is compact (and P^1 is easily seen to be homeomorphic to S^1), it follows by induction on n that P^n is metrizable for all n. (However, in this case the same conclusion could have been reached directly from the description of P^n as a quotient of S^n, by observing that its topology is induced by the metric which takes the shortest distance between representatives of equivalence classes, i.e.

$$d([\mathbf{x}], [\mathbf{y}]) = \min\{d(\mathbf{x}, \mathbf{y}), d(\mathbf{x}, -\mathbf{y})\}\ .)$$

Example 2. A particular case of glueing spaces together occurs when the subspace Z which is the domain of the glueing map f is just a single point $\{y_0\}$; then $X \cup_f Y$ is the quotient of the disjoint union obtained by making the single identification of $x_0 = f(y_0)$ with y_0. We call this the *wedge union* of X and Y, and denote it by $X \vee Y$. (This notation is ambiguous, since it fails to indicate *which* point of X is being identified with which point of Y; but it often doesn't matter. For example, if X and Y are both copies of S^n, then any point looks like any other point.) It is clear that if X and Y are subeuclidean then so is $X \vee Y$; if $X \subseteq \mathbf{R}^n$ and $Y \subseteq \mathbf{R}^m$ (and the points to be identified have been shifted to the origin in each case), then $X \vee Y$ is homeomorphic to the subset $(X \times \{\mathbf{0}\}) \cup (\{\mathbf{0}\} \times Y)$ of $\mathbf{R}^{n+m}$.

Example 3. Another example which we shall meet is that of the *cone* CX on a topological space X; this is defined as the quotient of the product $X \times I$ (where I is the closed unit interval $[0,1] \subseteq \mathbf{R}$) by the equivalence relation whose equivalence classes are $X \times \{1\}$ and all singletons $\{(x,t)\}$, $t < 1$; equivalently, it is the space $\{*\} \cup_f (X \times I)$, where $\{*\}$ is the one-point space and f is the unique map $(X \times \{1\}) \to \{*\}$. By the result quoted above, if X is compact and metrizable then CX is metrizable; if X is a compact subspace of $\mathbf{R}^n$, then CX may be identified with the subspace of $\mathbf{R}^{n+1}$ obtained by 'joining X to a point', i.e. to the set of all points on (closed) line segments joining points $(\mathbf{x}, 0)$ ($\mathbf{x} \in X$) to $(\mathbf{0}, 1)$ (or to any other point outside the hyperplane $\mathbf{R}^n \times \{0\}$). [Warning: for a non-compact subeuclidean space X, CX (defined as a quotient, as above) is **not** homeomorphic to the space obtained by joining X to a point; indeed, it is not even metrizable in general. *Exercise for the ambitious*: prove that $C\mathbf{R}$ is not metrizable.]

Notes on Abstract and Geometric Complexes

The distinction between abstract and geometric simplicial complexes is often a source of confusion; these notes are an attempt to clarify it.

An *abstract simplicial complex* is a purely combinatorial object: it is a pair (V, Σ) where V is a finite set (whose elements are called *vertices*) and Σ is a set of nonempty subsets of V, satisfying (i) every member of V occurs as a member of some $\sigma \in \Sigma$, and (ii) $\sigma \in \Sigma$ and $\emptyset \neq \tau \subseteq \sigma$ imply $\tau \in \Sigma$. A *simplicial map* $(V, \Sigma) \to (V', \Sigma')$ is a mapping $f \colon V \to V'$ such that $\{a_0, a_1, \ldots, a_n\} \in \Sigma$ implies $\{f(a_0), f(a_1), \ldots, f(a_n)\} \in \Sigma'$. As they stand, these definitions clearly have no topological content, and it is possible to develop the theory of simplicial complexes and their homology as a piece of pure combinatorics. (Indeed, the theory has applications in contexts other than the topological one which we present below.)

Example. The *complete n-simplex* Δ^n is the abstract complex with $V = \{0, 1, 2, \ldots, n\}$ and Σ taken to be the set of all nonempty subsets of V.

However, we are interested in abstract complexes only as a means to an end, the end being the convenient handling of a class of subeuclidean spaces called *polyhedra*. To introduce these, we define a *geometric simplicial complex* to be a finite set K of (geometric) simplices, all lying in the same Euclidean space $\mathbf{R}^n$, such that (i) if $\sigma \in K$ and τ is a face of σ, then $\tau \in K$, and (ii) if $\sigma, \tau \in K$, then $\sigma \cap \tau$ is (either empty or) the simplex spanned by the common vertices of σ and τ. It is clear that any geometric complex K gives rise to an *underlying abstract complex* (V, Σ), where V is the set of 0-simplices of K and Σ is the set of those finite subsets of V which span simplices of K (so that Σ is in bijective correspondence with K—and in practice we generally identify them).

By a *geometric realization* of an abstract simplicial complex (V, Σ), we mean a geometric complex whose underlying abstract complex (defined as above) is isomorphic to (V, Σ). Such a realization is specified by a map $r : V \to \mathbf{R}^n$ such that (i) for each $\sigma = \{a_0, \ldots, a_n\} \in \Sigma$, the points $r(a_0), \ldots, r(a_n)$ are affinely independent (and so span a simplex $r(\sigma)$), and (ii) for any two elements σ and τ of Σ, the intersection of $r(\sigma)$ and $r(\tau)$ is (no larger than) the simplex $r(\sigma \cap \tau)$ (to be interpreted as the empty set if $\sigma \cap \tau = \emptyset$).

For example, the (abstract) complete n-simplex Δ^n may be realized by the (geometric) complete n-simplex (also denoted Δ^n!), which consists of the n-simplex in $\mathbf{R}^n$ with vertices $\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ (where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbf{R}^n$) together with all its faces. More generally, one way to obtain a geometric realization of an arbitrary abstract complex (V, Σ) with $\operatorname{card} V = n + 1$ is to regard it as a subcomplex of Δ^n, i.e. to map the elements of V to the standard affinely independent set indicated above (but this is not the most efficient way; cf. question 2 on example sheet 2).

The *polyhedron* of a geometric simplicial complex K is the subspace $|K|$ of $\mathbf{R}^n$ which is the union of all the simplices of K. We say a space is *triangulable* if it is homeomorphic to a polyhedron; more precisely, a *triangulation* of a given space X is a pair (K, h) where K is a (geometric) simplicial complex and $h \colon X \to |K|$ is a homeomorphism. (However, in practice we generally omit any mention of h and say 'K is a triangulation of X'.)

If K and L are geometric simplicial complexes, a simplicial map f between their underlying abstract complexes induces a continuous map $|f|: |K| \to |L|$ by 'linear interpolation': that is, if $\mathbf{x} = t_0\mathbf{a}_0 + \cdots + t_n\mathbf{a}_n$ is a point in a simplex σ of K with vertices $\mathbf{a}_0, \ldots, \mathbf{a}_n$, then $|f|(\mathbf{x}) = t_0 f(\mathbf{a}_0) + \cdots + t_n f(\mathbf{a}_n)$ (which makes sense because the $f(\mathbf{a}_i)$ span a simplex of L). Clearly $|f|$ is continuous on each simplex of K, and the definitions on different simplices agree where they overlap; so $|f|$ is continuous on $|K|$. In practice we usually write $|f|$ as f; when we say '$f: |K| \to |L|$ is a simplicial map', we mean that f is the continuous map induced (as above) by a simplicial map between the underlying abstract complexes of K and L (the latter is of course uniquely determined by f, since it is the restriction of f to those points of $|K|$ which are vertices of K).

It is clear that if M is another geometric simplicial complex and $g: L \to M$ another simplicial map, then $|gf| = |g| \circ |f|: |K| \to |M|$. Hence in particular, if f is an isomorphism of abstract simplicial complexes, then $|f|$ is a homeomorphism. Thus *any two geometric realizations of a given abstract simplicial complex are homeomorphic.* This is the key result which tells us that the topological structure of a polyhedron (or more generally, of a triangulable space) is entirely recoverable from the combinatorial structure of the abstract simplicial complex underlying any triangulation of it.

However, not every continuous map between polyhedra is a simplicial map. The *Simplicial Approximation Theorem* tells us that every continuous map $f: |K| \to |L|$ is homotopic to a simplicial map, not necessarily from K to L but from some subdivision of K to L. The notion of *subdivision* may be defined very easily for abstract complexes: the (first) subdivision of a complex (V, Σ) is the complex (V', Σ'), where $V' = \Sigma$ and Σ' is the set of all those nonempty subsets of Σ which are totally ordered by inclusion. If K is a geometric realization of (V, Σ), then there is a geometric realization K' of (V', Σ') with the same polyhedron, obtained by mapping a simplex $\sigma = \{a_0, \ldots, a_n\}$ of (V, Σ) (regarded as a vertex of (V', Σ')) to the *barycentre* $\widehat{r(\sigma)} = (r(a_0) + \cdots + r(a_n))/(n+1)$ of the (geometric) simplex $r(\sigma)$. (Of course, it requires proof that this is a geometric realization, i.e. that the images of the simplices in Σ' don't overlap more than they should; and that every point of $|K|$ lies in some simplex of K'—the converse is easy.) By iterating this process, we define the nth (barycentric) subdivision $K^{(n)}$ of K for all $n \geq 0$: $K^{(0)} = K$, $K^{(n+1)} = (K^{(n)})'$.

Note that there are plenty of simplicial maps $f: K' \to K$; we get one by choosing, for each vertex $\hat{\sigma}$ of K', a vertex $f(\hat{\sigma})$ of σ. Moreover, any f constructed in this way induces a continuous map $|K'| \to |K|$ homotopic to the identity. But in general there are no simplicial maps $K \to K'$ whose underlying continuous maps are homotopic to the identity; thus the set of homotopy classes of maps $|K| \to |L|$ which contain simplicial maps $K \to L$ is a subset, and in general a proper subset, of the set of homotopy classes which contain simplicial maps $K' \to L$, and so on. The Simplicial Approximation Theorem says that, if we iterate this process far enough, we can eventually capture the homotopy class of any given map $|K| \to |L|$.