Algebraic Topology.

Introduction.

Topology: spaces and continuous functions (maps).
Spaces X and Y are homeomorphic ($X \cong Y$) if $\exists$ maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that $gf = 1_X$, $fg = 1_Y$.

Examples of Spaces: $\mathbb{R}^n$: n-tuples.
B^n : n-ball $= \{x \in \mathbb{R}^n : \|x\| < 1\}$
S^{n-1} : $n-1$-sphere $= \{x \in \mathbb{R}^n : \|x\| = 1\}$.
I : unit interval $= \{x \in \mathbb{R} : 0 \leq x \leq 1\}$
Torus : $S^1 \times S^1 =$  Annulus: $S^1 \times I =$ 

Möbius band $=$  $\subset \mathbb{R}^3$. Klein bottle.

For the course, metric spaces will do.

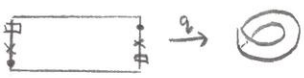
Recall: if X is a metric space, $U \subset X$, U is open iff for each $x \in U$ $\exists \delta > 0$ such that
$d(x, \tilde{x}) < \delta \Rightarrow \tilde{x} \in U$
$f : X \to Y$ is continuous iff for every open set $V \subset Y$, $f^{-1}V$ is open in X.

Definition: A topological space is a set X together with a collection of subsets of X that are called open, such that (i) ϕ and X are open,
(ii) any union of open sets is open
(iii) if U_1 and U_2 are open, so is $U_1 \cap U_2$.

Metric $\Rightarrow$ topological. ($\Leftarrow$ is false).

If X, Y are topological spaces, $X \times Y$ is the set of pairs (x, y). And, a set is open iff it is a union of sets $U \times V$, with U open in X, V open in Y.

Quotients: Suppose X is a topological space, $\sim$ an equivalence relation on set X.
Let $X/\sim$ be the set of equivalence classes. Have quotient map, $q : X \to X/\sim$, $x \mapsto [x]$.
Define quotient topology on $X/\sim$ by: V open in $X/\sim \Leftrightarrow q^{-1}V$ open in X.

Example:  - have topology on Möbius band.

Real projective n-space, $\mathbb{RP}^n = S^n /_{x \sim y \Leftrightarrow x = \pm y}$.

Note: $X/\sim$ may not be a metric space, even if X is.

Aim: associate groups with topological spaces: X, space $\downarrow f$, map Y, space $\longleftrightarrow$ $\pi_1(X)$, fundamental group of X $\downarrow f_*$, homomorphism $\pi_1(Y)$.

$\pi_1(X) \not\cong \pi_1(Y) \Rightarrow X \not\cong Y$.

Similarly, $H_r(X)$, rth homology group of X.
$$\begin{array}{c} H_r(X) \\ \downarrow - f_*, \text{ group homomorphism.} \\ H_r(Y) \end{array}$$

Classify surfaces. For example:  $\not\cong$ 

Brouwer fixed point theorem: any map $f: B^n \to B^n$ has a fixed point.

Also,  $\not\cong$ 

1. Homotopy and the Fundamental Group.

Definition: A homotopy between maps $f, g: X \to Y$ is a map $F: X \times I \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$ $\forall x \in X$. (Let $F_t: X \to Y$ be $F_t(x) = F(x,t)$. $F_0 = f$, $F_1 = g$)
Write $f \underset{F}{\simeq} g$.
If $A \subset X$ and $F(a,t) = f(a) = g(a)$ $\forall a \in A, t \in I$, then the homotopy is "rel A"

Lemma 1.1: "Homotopy rel A" is an equivalence relation on maps $X \to Y$.
Proof: (i) $f \simeq f$: Set $F(x,t) = f(x)$ $\forall t$.
(ii) If $f \underset{F}{\simeq} g$ rel A, define $G(x,t) = F(x, 1-t)$. Then $g \underset{G}{\simeq} f$.
(iii) If $f \underset{F}{\simeq} g$ rel A, $g \underset{G}{\simeq} h$ rel A, define $H: X \times I \to Y$ by $H(x,t) = \begin{cases} F(x,2t), & 0 \le t \le \frac{1}{2} \\ G(x,2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$ Then, $f \underset{H}{\simeq} h$.

Lemma 1.2: Suppose $X \overset{f_0}{\underset{f_1}{\rightrightarrows}} Y \overset{g_0}{\underset{g_1}{\rightrightarrows}} Z$, and $f_0 \underset{F}{\simeq} f_1$, $g_0 \underset{G}{\simeq} g_1$. Then $g_0 f_0 \simeq g_1 f_1$.
Proof: $g_0 f_0 \underset{g_0 F}{\simeq} g_0 f_1 \underset{G(f_1 \times 1_I)}{\simeq} g_1 f_1$. If F and G are rel A, then the answer is rel A.

Lemma 1.3 (Linear homotopy): Suppose $f, g: X \to Y \subset \mathbb{R}^n$ and $\forall x \in X$ the straight line from $f(x)$ to $g(x)$ is in Y. Then $f \simeq g$. If $f|_A = g|_A$ for some $A \subset X$, then $f \simeq g$ rel A.
Proof: Define $F(x,t) = (1-t) f(x) + t g(x)$

Corollary: Suppose $f, g: X \to S^{n-1}$ and $f(x) \neq -g(x)$ $\forall x$. Then $f \simeq g$.
Proof:
$$\begin{array}{lcl} X \xrightarrow{f,g} S^{n-1} & & \\ X \xrightarrow{f,g\ \simeq} \mathbb{R}^n - \{0\} & \longrightarrow & S^{n-1} \\ \qquad\qquad a & \longmapsto & \frac{a}{\|a\|} \end{array}$$
– and use lemma 1.2.

Example: $f: X \to S^{n-1}$ and $\exists\, a \in S^{n-1}$ with $f(x) \neq a$ $\forall x \in X$. Then $f \simeq$ constant map. (Take $g(x) = -a$ $\forall x$)

Definition: Spaces X and Y are homotopy equivalent, $X \simeq Y$, if $\exists$ maps $X \overset{f}{\underset{g}{\rightleftarrows}} Y$ such that $gf \simeq 1_X$, $fg \simeq 1_Y$. (Say f is a homotopy equivalence, with homotopy inverse g).

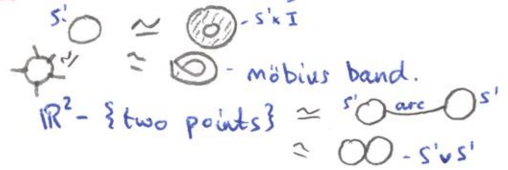
Note: homeomorphism $\Rightarrow$ homotopy equivalence.

Lemma 1.4: "$\simeq$" is an equivalence relation on class of topological spaces.
Proof: Transitivity: $X \overset{f}{\underset{g}{\rightleftarrows}} Y \overset{\varphi}{\underset{\psi}{\rightleftarrows}} Z$. $g\psi\varphi f \simeq gf$ (lemma 1.2), $\simeq 1_X$ (lemma 1.1)

Definition: X is contractible if $X \simeq \{\text{a single point}\}$

Definition: Suppose $A \subset X$ and $r: X \to X$ such that $r(X) \subset A$, $r(A) = 1_A$. Then r is a retraction.
If $r \simeq 1_X$ (rel A), r is a (strong) deformation retraction of X to A.
$X \underset{i}{\overset{r}{\rightleftarrows}} A$, $ri = 1_A$, $ir = r \simeq 1_X$, so X and A are homotopy equivalent.

Examples: $\{0\}$ is a strong deformation retract of $\mathbb{R}^n$ (lemma 1.3).



Definition: A path in X from x_0 to x_1 is a map $u: I \to X$ such that $u(0) = x_0$, $u(1) = x_1$.
If $x_0 = x_1$, the path is a loop at x_0.

Definition: Suppose $u_1, \dots, u_n$ are paths in X, with u_i from x_{i-1} to x_i.
Define a product path by: $(u_1 \cdot \dots \cdot u_n)(s) = u_i(ns - i + 1)$, $\frac{i-1}{n} \le s \le \frac{i}{n}$, $s \in I$.
Define an inverse path by: $u^{-1}(s) = u(1-s)$, $s \in I$. Note: $(u_1 \cdot u_2)^{-1} = u_2^{-1} \cdot u_1^{-1}$.

Lemma 1.5: (i) Suppose $u_i \underset{F_i}{\simeq} v_i$ rel ∂I $(= \{0,1\})$, u_i, v_i paths from x_{i-1} to x_i. Then,
$u_1 \cdot \dots \cdot u_n \simeq v_1 \cdot \dots \cdot v_n$ rel ∂I.
(ii) Suppose $u \underset{F}{\simeq} v$ rel ∂I. Then $u^{-1} \simeq v^{-1}$ rel ∂I.

Proof: (i) Let $F(s,t) = F_i(ns + 1 - i, t)$, $\frac{i-1}{n} \le s \le \frac{i}{n}$.
(ii) Exercise.

Note: Any two paths in I are homotopic rel ∂I if they agree on ∂I.

Lemma 1.6: (i) If u_i is a path in X from x_{i-1} to x_i, $i = 1, \dots, n$, then $(u_1 \cdot \dots \cdot u_r) \cdot (u_{r+1} \cdot \dots \cdot u_n) \simeq u_1 \cdot \dots \cdot u_n$ rel ∂I.
(ii) If u is a path in X from x_0 to x_1 and e_0 and e_1 are constant x_0 and x_1 respectively, then $e_0 \cdot u \simeq u$ rel ∂I, and $u \cdot e_1 \simeq u$ rel ∂I.
(iii) $u \cdot u^{-1} = e_0$ rel ∂I, $u^{-1} \cdot u \simeq e_1$ rel ∂I.

Proof: (i) $(\text{LHS})(s) = (\text{RHS})\,\varphi(s)$, where $\varphi: I \to I$,

$$\begin{aligned} \varphi(0) &= 0 \\ \varphi(\tfrac{1}{2}) &= \tfrac{r}{n} \\ \varphi(1) &= 1 \end{aligned} \quad \begin{aligned} &\text{linear in between, } \varphi(s) = \tfrac{2rs}{n} \\ &\text{linear in between, } \varphi(s) = 2s - 1 + \tfrac{2r}{n}(1-s) \end{aligned}$$

But $\varphi \simeq 1_I$ rel ∂I, so by lemmas 1.3 and 1.2, LHS $\simeq$ RHS rel ∂I.

(ii) $(e_0 \cdot u)(s) = u(\varphi(s))$, where $\varphi: I \to I$, $\varphi(s) = \begin{cases} 0, & s \le 1/2 \\ 2s-1, & \frac{1}{2} \le s \le 1 \end{cases}$

$\varphi \simeq 1_I$ rel ∂I, by linear homotopy. So, $u \simeq u\varphi$ rel ∂I, so $e_0 \cdot u \simeq u$ rel ∂I.

(iii) $(u \cdot u^{-1})(s) = u(\varphi(s))$, where $\varphi(s) = \begin{cases} 2s, & 0 \le s \le 1/2 \\ 2(1-s), & \frac{1}{2} \le s \le 1. \end{cases}$

φ is homotopic to the constant map at 0 rel ∂I, so $u \cdot u^{-1} = u\varphi \simeq e_0$ rel ∂I.

Theorem 1.7: The set of homotopy classes rel ∂I of loops based at $x_0 \in X$ forms a group $\pi_1(X, x_0)$, the fundamental group of X with base point x_0, where if u and v are loops at x_0 and $[u]$ is homotopy class rel ∂I, then $[u][v] = [u.v]$.

Proof: Product is well-defined, by lemma 1.5 (i). Define $[u]^{-1} = [u^{-1}]$, well-defined by lemma 1.5 (ii). Associativity - from lemma 1.6 (i). Define identity $e = [e_0]$, okay by lemma 1.6 (ii). $[u][u]^{-1} = [u][u^{-1}] = [u.u^{-1}] = [e_0] = e$, using lemma 1.6 (iii).

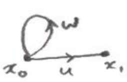
Facts: $\pi_1(S^1, x_0) \cong \mathbb{Z}$. Generators, $[0,1] \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$; $s \mapsto e^{2\pi i s}$.
$\pi_1(S^2, x_0) \cong \{1\}$, the trivial group.
$\pi_1(\mathbb{RP}^2, x_0) \cong \mathbb{Z}/2\mathbb{Z}$, the group of two elements. $\pi_1(\text{torus}) \cong \mathbb{Z} \oplus \mathbb{Z}$.
$\pi_1(\mathbb{R}^n) \cong \{1\}$. $\pi_1(\text{Klein bottle})$ is not abelian. (Proofs later).

Theorem 1.8: A map $f: X, x_0 \to Y, y_0$ induces a group homomorphism $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ such that: (i) $f \simeq f'$ rel $\partial I \Rightarrow f_* = f'_*$,
(ii) $(1_X)_*$ is the identity homomorphism,
(iii) if $g: Y, y_0 \to Z, z_0$, then $(gf)_* = g_* f_* : \pi_1(X, x_0) \to \pi_1(Z, z_0)$.

Proof: Define $f_*[u] = [fu]$, well-defined by lemma 1.2. Then, $f_*([u][v]) = f_*[u.v] = [f(u.v)]$ $= [(fu).(fv)] = [f(u)][f(v)] = (f_*[u])(f_*[v])$. (i), (ii), (iii) are immediate.

Aside: Category: objects and maps. Spaces, continuous functions $\xrightarrow{\text{functor}}$ groups, homomorphisms.
Homeomorphic spaces $\Rightarrow$ isomorphic groups.
$X \underset{g}{\overset{f}{\rightleftarrows}} Y$, $fg = 1_Y$, $gf = 1_X$, so $f_* g_* = (1_Y)_* = 1_{\pi_1(Y, y_0)}$, $g_* f_* = (1_X)_* = 1_{\pi_1(X, x_0)}$ $\Rightarrow f_*, g_*$ are isomorphisms.

Theorem 1.9: A path u in X from x_0 to x_1 induces an isomorphism $u_\#: \pi_1(X, x_0) \to \pi_1(X, x_1)$ such that
(i) $u \simeq \hat{u}$ rel $\partial I \Rightarrow u_\# = \hat{u}_\#$,
(ii) $e_{0\#}$ = identity,
(iii) if v is a path from x_1 to x_2, then $(u.v)_\# = v_\# u_\#$,
(iv) if $f: X, x_0, x_1 \to Y, y_0, y_1$, then $(fu)_\# f_* = f_* u_\#$

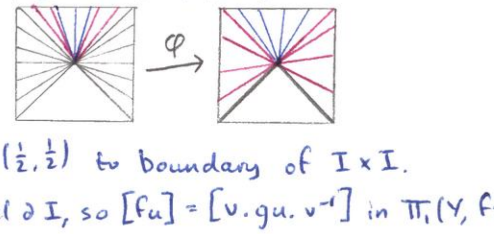


Proof: If w is a loop in X at x_0, define $u_\#[w] = [u^{-1}.w.u]$ - well-defined:
$u_\#([v][w]) = u_\#([v.w]) = [u^{-1}.v.w.u] = [u^{-1}.v.u.u^{-1}.w.u] = [u^{-1}.v.u][u^{-1}.w.u] = u_\#[v]\, u_\#[w]$. (i), (ii), (iii) easy now.
(iv) $f_* u_\#[w] = [f(u^{-1}.w.u)] = [(fu)^{-1}.(fw).(fu)] = (fu)_\#(f_*[w])$
$u_\#(u^{-1})_\# = (u^{-1}.u)_\# = e_{1\#} = 1$, and $(u^{-1})_\# u_\# = 1$, similarly. So $u_\#$ is an isomorphism.

If X is path-connected, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ always. Write $\pi_1(X)$.
X "simply connected" $\equiv$ path-connected and $\pi_1(X)$ the trivial group.

Theorem 1.10: Suppose $f \underset{F}{\simeq} g : X \to Y$. Suppose $x_0 \in X$ and v is the path from $f(x_0)$ to $g(x_0)$, defined by $v(t) = F(x_0, t)$. Then, $v_\# f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, g(x_0))$.

Proof: Let u be a loop at x_0. Define $\varphi: I \times I \to I \times I$:



$\varphi(s, 0) = (s, 0)$, $\varphi(0, t) = (0, 0)$, $\varphi(1, t) = (1, 0)$,
$\varphi(s,1) = \begin{cases} (0, 3s), & 0 \le s \le 1/3 \\ (3s-1, 1), & 1/3 \le s \le 2/3 \\ (1, 3-3s) & 2/3 \le s \le 1 \end{cases}$, $\varphi(\tfrac{1}{2}, \tfrac{1}{2}) = (\tfrac{1}{2}, \tfrac{1}{2})$
φ linear on segments from $(\tfrac{1}{2}, \tfrac{1}{2})$ to boundary of $I \times I$.
$F(u \times 1)\varphi : I \times I \to Y$ is a homotopy. $fu \simeq v.gu.v^{-1}$ rel ∂I, so $[fu] = [v.gu.v^{-1}]$ in $\pi_1(Y, fx_0)$.
$\Rightarrow f_*[u] = (v^{-1})_\# g_*[u]$, so $v_\# f_* = g_*$.

Corollary: Let $f: X \to Y$ be a homotopy equivalence and $f(x_0) = y_0$, then $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

Proof: Given $X \underset{g}{\overset{f}{\rightleftarrows}} Y$, $gf \simeq 1_X$, $fg \simeq 1_Y$. For some path v from $gf(x_0)$ to x_0,

$$v_{\#}(gf)_* = (1_X)_* = 1_{\pi_1(X, x_0)}.$$

So $(gf)_*$ is an isomorphism also.

isomorphism, by Theorem 1.9   Theorem 1.10   isomorphism

$$\pi_1(X, x_0) \xrightarrow[f_*,\ \text{injective}]{} \pi_1(Y, fx_0) \xrightarrow[g_*,\ \text{surjective}]{} \pi_1(X, gf(x_0))$$

Similarly, $\pi_1(Y, f(x_0)) \xrightarrow[g_*,\ \text{injective}]{} \pi_1(X, gf(x_0)) \xrightarrow[f_*,\ \text{surjective}]{} \pi_1(Y, fgf(x_0))$

Therefore, $g_*: \pi_1(Y, f(x_0)) \to \pi_1(X, gf(x_0))$ is an isomorphism.

$g_* f_*$ is an isomorphism, so f_* is an isomorphism.

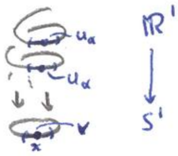
2. Covering Spaces, Covering Maps.

Example: $p: \mathbb{R}^1 \to S^1$; $t \mapsto e^{2\pi i t}$. Wish p^{-1} existed.

Note: For this chapter, X will be path-connected.

Definition: A covering space $\tilde{X}$ of X with covering map $p: \tilde{X} \to X$ is a non-empty path-connected space $\tilde{X}$ such that, for each $x \in X$ $\exists$ open V in X, $x \in V$, with the property that $p^{-1}V$ is the disjoint union $\bigcup_\alpha U_\alpha$ of open subsets $U_\alpha \subset \tilde{X}$ and $p|_{U_\alpha}: U_\alpha \to V$ is a homeomorphism for each α.

Example: For p in above example:



This definition implies that p is surjective, and maps open sets to open sets.

Examples: (i) $\mathbb{R}^1 \xrightarrow{p} S^1$. (ii) $X \xrightarrow{1} X$. (iii) $S^1 \to S^1$, $p(z) = z^n$, fixed n. (iv) $p: S^n \to \mathbb{RP}^n \equiv {}^{x}/_{x \sim \pm x}$

(v) $\hat{p}: S^3 \to L_{p,q}$ - Lens space, p, q coprime integers. $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$.
Define $g: S^3 \to S^3$; $g(z_1, z_2) = (e^{2\pi i/p} z_1, e^{-2\pi i q/p} z_2)$. g generates an action of $G \equiv \mathbb{Z}/p\mathbb{Z}$. $L_{p,q} \equiv S^3/G$, ie, $x \sim y \iff \exists\, g^r$ such that $g^r x = y$.
$L_{p,q}$ is a 3-manifold. $\hat{p}$ is a quotient map.

Lebesgue's Lemma: Let X be a compact metric space and suppose that X is the union of a collection $\{U_i\}$ of open subsets. Then there exists a real number $\delta > 0$ such that if $S \subset X$ and S has diameter less than δ, then S is contained in one of the U_i.

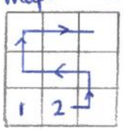
⊛ **Proof:** Any $x \in X$ is in some U_i so choose $\varepsilon(x) > 0$ so that the open ball $B(x, 2\varepsilon(x))$ is contained in this U_i. Clearly $X = \bigcup_{x \in X} B(x, \varepsilon(x))$, so by compactness $X = \bigcup_{i=1}^{n} B(x_i, \varepsilon(x_i))$, for some finite set $\{x_1, .., x_n\} \subset X$. Let $\delta = \min\{\varepsilon(x_1), .., \varepsilon(x_n)\}$ and suppose S has diameter less than this δ. If $y \in S$, then $y \in B(x_j, \varepsilon(x_j))$ for some j, and so $S \subset B(x_j, 2\varepsilon(x_j))$ which is, by construction, contained in some U_i. ⊛

Lemma 2.1 (Path Lifting Property): Suppose $p: \tilde{X} \to X$ is a covering map. Suppose $u: I \to X$ is a path, and $\tilde{x}_0 \in \tilde{X}$ is such that $p\tilde{x}_0 = u(0)$. Then $\exists$ a unique path $\tilde{u}: I \to \tilde{X}$ such that $\tilde{u}(0) = \tilde{x}_0$ and $p\tilde{u} = u$.

Proof: $\{V$, as in definition of cover$\}$ are an open-covering of X. (So $X = \cup V$). I is compact, so by Lebesgue's lemma on $\{u^{-1}(V)\}$, $\exists$ dissection of I: $0 = t_0 < t_1 < \dots < t_n = 1$ such that $u[t_{i-1}, t_i] \subset V_i$, some V_i, some V as in definition. Assume $\tilde{u}$ is defined on $[0, t_{i-1}]$. $\tilde{u}(t_{i-1}) \in U_{i,\alpha}$, some open $U_{i,\alpha} \subset \tilde{X}$, and $p: U_{i,\alpha} \to V_i$ is a homeomorphism (by definition). Define $\tilde{u}|_{[t_{i-1}, t_i]} = (p|_{U_{i,\alpha}})^{-1} u$. Continue in this way.
Uniqueness: suppose $\hat{u}$ is another lift, $\hat{u}(0) = \tilde{x}_0$. Let $\tau = \sup\{t : \hat{u}|_{[0,t]} = \tilde{u}|_{[0,t]}\}$. (Note that $\hat{u}(\tau) = \tilde{u}(\tau)$, by continuity). If $\tau < 1$: $u(\tau) \in$ some V as in definition, so $\hat{u}(\tau)$ and $\tilde{u}(\tau)$ both $\in$ some open U_α as in definition. For sufficiently small δ, $\hat{u}(\tau+\delta)$ and $\tilde{u}(\tau+\delta) \in U_\alpha$ by continuity. So, $\hat{u}(\tau+\delta) = \tilde{u}(\tau+\delta)$, as $p|_{U_\alpha}$ is injective. $\divideontimes$. So $\tau = 1$.

Lemma 2.2 (Homotopy Lifting Property): Suppose $p: \tilde{X} \to X$ is a covering map. Suppose we have $F: I \times I \to X$ and $\tilde{F}: I \times \{0\} \to \tilde{X}$ such that $p\tilde{F}(s,0) = F(s,0)$ $\forall s$. Then, $\exists$ unique extension of $\tilde{F}$ to $I \times I \xrightarrow{\tilde{F}} \tilde{X}$ so that $p\tilde{F} = F$.

Proof: Dissect $I \times I$ into small squares, so $F(\text{square}) \subset$ some V as in definition of covering map. Order squares "snakes and ladders" fashion, so that $(\text{square } i) \cap \{\text{predecessors}\}$ is connected. Proceed as in lemma 2.1.



Uniqueness follows from ~~so~~ lemma 2.1, as $\tilde{F}|_{s \times I}$ is a lift of $F|_{s \times I}$, unique.

$\{$Homeomorphisms: $S \to S$, S a space$\}$ is a group, but vast. If a group G is (isomorphic to) a subgroup of this, say G acts on S. Define $S/G = S/_{s_1 \sim s_2 \Leftrightarrow s_1 = g s_2,\text{ some } g \in G}$.

Theorem 2.3: Suppose a group G acts as a group of homeomorphisms on a simply connected space Y, and each $y \in Y$ has an open neighbourhood U such that $U \cap gU = \emptyset$ $\forall g \in G - \{1\}$. Then $\pi_1(Y/G) \cong G$.

Proof: Consider quotient $p: Y \to Y/G$. If $y \in Y$, then $y \in$ open U such that $U \cap gU = \emptyset$, $g \neq 1$. gU is open (as g is a homeomorphism), so pU is open, (as $p^{-1}(pU) = \bigcup_{g \in G} gU$, a union of open sets). $p: U \to pU$ is an injection, as $U \cap gU = \emptyset$, $g \neq 1$. If V open and $V \subset U$, pV is open (as above). $\therefore$ $p|_U: \underset{gU}{\overset{U}{}} \rightarrow pU$ is a homeomorphism. So p is a covering map.
Fix $y_0 \in Y$. Let $g \in G$. Define $\varphi: G \to \pi_1(Y/G, p(y_0))$ as follows:
Let u be a path from y_0 to gy_0 in Y. Let $\varphi g = [pu]$, a loop at py_0, $\in \pi_1(Y/G, py_0)$. If $u \simeq v$ rel ∂I, then $[pu] = [pv]$. But Y is simply connected (so any $u \simeq v$ rel ∂I), so φ is well-defined. φ is surjective by lemma 2.1 and injective by lemma 2.2 (detail - exercise).
If u is a path from y_0 to $g_1 y_0$ and v a path from y_0 to $g_2 y_0$, then $g_1 v$ is from $g_1 y_0$ to $g_1 g_2 y_0$, and $u \cdot (g_1 v)$ is from y_0 to $g_1 g_2 y_0$.
$pg_1 v = pv$ (as p is a quotient map), so $[pu][pv] = [p(u \cdot g_1 v)]$, so $(\varphi(g_1))(\varphi(g_2)) = \varphi(g_1 g_2)$

Corollary: $\pi_1 S^1 \cong \mathbb{Z}$, $\pi_1(L_{p,q}) \cong \mathbb{Z}/p\mathbb{Z}$, $\pi_1(\text{torus}) \cong \mathbb{Z} \oplus \mathbb{Z}$.
$\mathbb{RP}^n = S^n/\text{action by } \mathbb{Z}/2\mathbb{Z}$. $\therefore$ $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$ $\forall n \geq 2$.

<u>Definition</u>: A map $f: S^n \to S^n$ is <u>antipodal</u> iff $\forall x \in S^n$, $f(-x) = -f(x)$.

<u>Theorem 2.4</u>: $\nexists$ antipodal map $S^2 \to S^1$

$$\begin{array}{ccc} f: S^2 & \to & S^1 \\ \downarrow q_2 & & \downarrow q_1 \end{array}$$

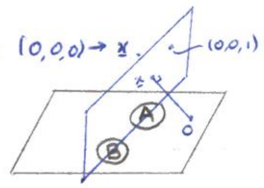
<u>Proof</u>: Suppose $f: S^2 \to S^1$ is antipodal. f induces $g: \mathbb{R}P^2 \to \mathbb{R}P^1$, q_i quotient.

$q_1 f = g q_2$. $q_{1*}: \pi_1 S^1 \to \pi_1(\mathbb{R}P^1)$
$\mathbb{Z} \to \mathbb{Z} \cong S^1$
$1 \mapsto 2$

Consider $I \xrightarrow{u} S^2 \xrightarrow{f} S^1$, $S^2 \xrightarrow{q_2} \mathbb{R}P^2$, $u(t) = (\cos \pi t, \sin \pi t, 0)$. fu is a path from some x to $-x$.
Let v be any (standard) path in S^1 from $-x$ to x. $[fu . v] \in \pi_1(S^1, x)$, $= n \in \mathbb{Z}$, say.
$[q_1 fu] = 2n \pm 1 \in \mathbb{Z} \cong \pi_1 \mathbb{R}P^1$, and $[q_1 fu] = q_{1*} f_* [u] = g_* q_{2*}[u]$.
But g_* is the zero map, $g_*(?) = 0 \in \mathbb{Z}$, as $g_*: \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$. $\therefore 2n \pm 1 = 0$ #.

<u>Corollary (Borsuk)</u>: If $f: S^2 \to \mathbb{R}^2$, continuous, then $\exists x \in S^2$ such that $f(x) = f(-x)$.
<u>Proof</u>: Suppose $f(x) \neq f(-x)$ $\forall x \in S^2$. Define $g(x) = \frac{f(-x) - f(x)}{\|f(-x) - f(x)\|}$, so $g: S^2 \to S^1$, antipodal. #.

<u>Corollary</u>: If A and B are bounded, measureable sets in $\mathbb{R}^2$, $\exists$ a straight line bisecting both A and B ("<u>Ham Sandwich Theorem</u>").



<u>Proof</u>: If $\underline{x} \in S^2$, let $\Pi_{\underline{x}}$ = plane through $(0,0,1)$ perpendicular to $(0,0,0) \to \underline{x}$.
$f_1(x) = \mu$ (part of A on same side of $\Pi_{\underline{x}}$ as is $\underline{x}$)
$f_2(x) = \mu$ (part of B on same side of $\Pi_{\underline{x}}$ as is $\underline{x}$).
Borsuk $\Rightarrow \exists x$ with $(f_1(x), f_2(x)) = (f_1(-x), f_2(-x))$.

* <u>Lemma 2.5</u>: Suppose $p: \tilde{X} \to X$ is a covering map. Take $\tilde{x}_0 \in \tilde{X}$, let $x_0 = p(\tilde{x}_0)$. The group homomorphism, $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective. [$p_* \pi_1(\tilde{X}, \tilde{x}_0)$ is called the <u>group of the covering</u>].
Right cosets of $p_* \pi_1(\tilde{X}, \tilde{x}_0)$ in $\pi_1(X, x_0) \xleftrightarrow{\text{bijection}} p^{-1} x_0$.
<u>Proof</u>: If u is a loop at x_0 and $pu \simeq$ constant rel ∂I, lift homotopy (lemma 2.2).
So p_* is injective.
Suppose $[v] \in \pi_1(X, x_0)$ and $a \in p^{-1} x_0$. Lift v to be a path in $\tilde{X}$ from a to some $b \in p^{-1} x_0$ (lemma 2.1). This gives a map: $\pi_1(X, x_0) \to$ permutations of $p^{-1} x_0$; $[v] \mapsto (a \mapsto b$ as above$)$.
[Note that permutations act on right]. Lemma 2.2 $\Rightarrow$ well-defined.
Stabiliser of $\tilde{x}_0$ is $p_* \pi_1(\tilde{X}, \tilde{x}_0)$. Orbit of $\tilde{x}_0$ is all $p^{-1} x_0$ ($\tilde{X}$ is path-connected).

<u>Lemma 2.6</u>: Suppose $p: \tilde{X} \to X$ is a covering map, $p\tilde{x}_0 = x_0$. Suppose Y is path-connected, and locally path-connected (ie, any point has arbitrarily small path-connected neighbours).
Let $y_0 \in Y$, suppose we have $f: Y, y_0 \to X, x_0$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "g"] \arrow[r, "f"'] & X \end{tikzcd}$$

Then, $\exists$ lift map $g: Y, y_0 \to \tilde{X}, \tilde{x}_0$ with $pg = f \Leftrightarrow f_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0)$.
Such a g is unique.
<u>"Proof"</u>: g exists $\Rightarrow p_* g_* = f_*$. This does "$\Rightarrow$".
Conversely, define g: let $u: I \to Y$, $u(0) = y_0$, $u(1) = a \in Y$. fu is a path in X.
Lift to $\hat{f}: I \to \tilde{X}$, $\hat{f}(0) = \tilde{x}_0$. Define $\hat{f}(1) = g(a)$ - forced definition.
Condition $\Rightarrow g$ is well-defined. Locally path-connected $\Rightarrow g$ is continuous.

Covers are equivalent if $\exists$ homeomorphism h:
$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "h"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}, \quad p_2 h = p_1.$$

Same group $\Rightarrow$ equivalent.
$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, dashed, bend left, "g_1"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[ll, dashed, "g_2"] \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$
(using lemma 2.6). $\Rightarrow$
$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, shift left, "1"] \arrow[rr, shift right, "g_2 g_1"'] \arrow[dr, "p_1"'] & & \tilde{X}_1 \arrow[dl, "p_1"] \\ & X & \end{tikzcd}$$
So $g_1 g_2 = 1$, $g_2 g_1 = 1$.

Theorem 2.7: If X is path-connected and locally contractible (ie, x has arbitrarily small contractible neighbourhoods), then X has a simply-connected covering $\hat{X} \to X$, unique by above. $\pi_1(X, x_0)$ acts on $\hat{X}$ (the universal cover of X) with quotient a copy of X.

Proof: A point of $\hat{X}$ is a homotopy class of paths in X starting at x_0.

Corollary: Suppose G is any subgroup of $\pi_1(X, x_0)$. Then G acts on $\hat{X}$ (trivially). $\pi_1(\hat{X}/G) \cong G$, by Theorem 2.3, ie $\exists$ covering of X with G as its fundamental group. "Have" same group, gives homeomorphic covers with base points. Forgetting base points leads to:
"conjugacy classes of subgroups of $\pi_1(X, x_0) \xleftrightarrow{\text{bijection}}$ coverings of X"

$$\begin{array}{c} \hat{X} \\ \downarrow \text{a covering.} \\ \hat{X}/G \\ \downarrow \text{a covering.} \\ \hat{X}/\pi_1(X, x_0) = X \end{array}$$

$*$.

3. Simplicial Complexes.

Definition: Points $a_0, a_1, \ldots, a_n \in \mathbb{R}^m$ are (affine) independent if vectors $a_1 - a_0, \ldots, a_n - a_0$ are linearly independent. $\left(\Leftrightarrow \sum_{i=0}^{n} \lambda_i a_i = 0 \text{ and } \sum_{i=0}^{n} \lambda_i = 0 \Rightarrow \lambda_i = 0 \ \forall i \right)$

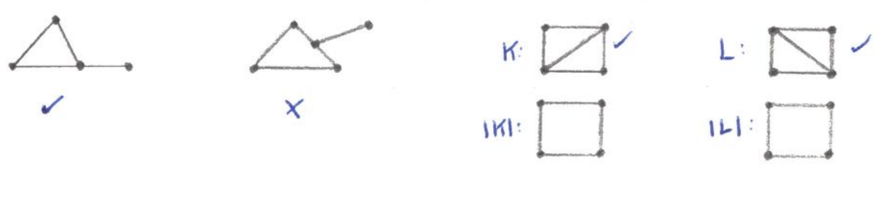
If this is so, the n-dimensional simplex σ^n with vectors $a_0, a_1, \ldots, a_n$ is: $\left\{ \sum_0^n \lambda_i a_i : \lambda_i \geq 0, \sum \lambda_i = 1 \right\}$.
Write $\sigma^n = (a_0 a_1 \ldots a_n)$. Examples: σ^0: $\bullet$, σ^1: , σ^2: , etc.

A simplex τ is a face of a simplex σ, $\tau \leq \sigma$, if $\{\text{vertices of } \tau\} \subseteq \{\text{vertices of } \sigma\}$.
The interior of σ, $\mathring{\sigma}$, is: $\sigma - \bigcup(\text{proper faces of } \sigma) = \left\{ \sum \lambda_i a_i : \lambda_i > 0, \sum \lambda_i = 1 \right\}$.
The barycentre of σ is: $\frac{1}{n+1}(a_0 + \cdots + a_n) \in \mathring{\sigma} \subset \sigma$.
σ^n is closed, convex, compact, connected and contractible.
$\sigma = \tau \Leftrightarrow$ vertices of σ = vertices of τ.

Definition: A (finite simplicial) complex K is a finite set of simplexes in some $\mathbb{R}^m$ such that:
(i) $\sigma \in K$, $\tau \leq \sigma \Rightarrow \tau \in K$
(ii) $\sigma, \tau \in K \Rightarrow \sigma \cap \tau = \phi$, or is a face of σ and τ.

$|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^m$ is the underlying polyhedron of K.

Examples:



The dimension of $K = \max\{\dim\sigma : \sigma \in K\}$

A triangulation of a space X is a homeomorphism $t : |K| \to X$ for some K.

[Example of a nasty space: Hawaiian earring,  - circles, radius $\frac{1}{n}$, centre $(\frac{1}{n}, 0)$]

A subcomplex L of K is a subset of simplexes of K satisfying (i). ((ii) follows)

Definition: Let K, L be simplicial complexes. A function $f : |K| \to |L|$ is a simplicial map (wrt K, L) if:
(i) for each simplex $(a_0 a_1 \ldots a_n) \in K$, $\{fa_0, \ldots fa_n\}$ are vertices of a simplex of L,
(ii) $f(\Sigma \lambda_i a_i) = \Sigma \lambda_i f(a_i)$
Note that f is continuous.

Lemma 3.1: $|K| = \bigsqcup_{\sigma \in K} \mathring{\sigma}$ ($\sqcup$ means 'disjoint union').

Proof: If $x \in \sigma \in K$, $x = \sum_0^n \lambda_i a_i$, $\sigma = (a_0 \ldots a_n)$, $x \in$ interior of face of σ with vertices $\{a_i\}$ for which $\lambda_i > 0$

Note: If v is a vertex, $\mathring{v} = v$.

Definition: If a is a vertex of K, the star of a, $\text{star}(a, K) = \bigcup_{a \in \sigma \in K} \mathring{\sigma}$. Example: 

Lemma 3.2: $\{\text{star}(a,K) : a$ is a vertex of $K\}$ is an open-cover of $|K|$.

Proof: If $\sigma \in K$, $\mathring{\sigma} \subset \text{star}$ (any vertex of σ) $\therefore \bigcup \mathring{\sigma} = \bigcup(\text{stars}) = |K|$, by lemma 3.1.
$|K| - \text{star}(a,K) = \bigcup_{\substack{\sigma \in K,\, a \text{ not} \\ \text{a vertex of } \sigma}} \sigma$ - a finite union of closed sets, so closed. $\therefore$ $\text{star}(a,K)$ is open in $|K|$.

Definition: A simplicial map $f : |K| \to |L|$ is a simplicial approximation to a (continuous) map $\varphi : |K| \to |L|$ if, for every vertex $a \in K$, $\varphi(\text{star}(a,K)) \subset \text{star}(f(a), L)$

Notes: $\varphi(a) \in \mathring{\sigma}$, some $\sigma \in L$, $f(a)$ being a vertex of L.
Composition of approximations is an approximation to a composition.

Lemma 3.3: Suppose $f : |K| \to |L|$ is a simplicial approximation to φ. Then $\varphi \simeq f$ along straight lines.
(rel A, where $A = \{x : f(x) = \varphi(x)\}$)

Proof: Suppose $x \in \mathring{\sigma}$, $\sigma \in K$, $\sigma = (a_0 \ldots a_n)$. $x \in \bigcap_{i=0}^{n} \text{star}(a_i, K)$ $\therefore \varphi(x) \in \bigcap \text{star}(f(a_i), L)$.
If $\varphi(x) \in \mathring{\tau}$, $\tau \in L$, then $f(a_i)$ is a vertex of τ. $\therefore f(x) \in \tau$.
$\therefore$ Line segment from $\varphi(x)$ to $f(x)$ $\subset \tau \subset |L|$. Then use linear homotopy.

Addendum: If $X \subset |K|$, $\varphi(X) \subset |M|$, M a subcomplex, then $f(X) \subset M$, and homotopy is in M.

Lemma 3.4: Suppose $\varphi : |K| \to |L|$ is continuous, and for each vertex $a_i \in K$, $\varphi(\text{star}(a_i, K)) \subset \text{star}(b_i, L)$ for some vertex $b_i \in L$. Then $\exists$ a simplicial approximation f to φ with $f(a_i) = b_i$.

Proof: Let $\sigma = (a_0 \ldots a_n) \in K$ (having relabelled). If $x \in \mathring{\sigma}$, $\varphi(x) \in \mathring{\tau}$, some $\tau \in L$, where $b_0, \ldots, b_n$ are vertices of τ (maybe $b_i = b_j$). Then $\{b_0, \ldots, b_n\}$ are vertices, not distinct, of some face of τ. $\therefore$ Can define $f(a_i) = b_i$, and extend linearly over a_i.

"Make stars of vertices smaller" - subdivision: 

If $\sigma = (a_0 \dots a_n)$, barycentre $\hat{\sigma} = \frac{1}{n+1}\sum_{i=0}^{n} a_i$

The boundary, $\partial\sigma = \{$ proper faces of $\sigma\}$ is a complex.

If K is a simplicial complex, the s-skeleton of $K = \{\sigma \in K : \dim\sigma \le s\}$.

(So, $\partial\sigma$ = $(n-1)$-skeleton of σ, where $\dim\sigma = n$)

Definition: If K is a complex, define $K^{(1)}$, the first derived subdivision of K inductively on $\dim K$: if $\dim K = 0$, $K^{(1)} = K$.

If $\dim K = n$, $K^{(1)} = \begin{cases} (K_{(n-1)})^{(1)} \leftarrow (n-1)\text{-skeleton} \\ \hat{\sigma} \text{ if } \dim\sigma = n \\ \hat{\sigma}\tau : \sigma \in K, \dim\sigma = n, \tau \in (\partial\sigma)^{(1)} \leftarrow \text{okay by induction} \end{cases}$

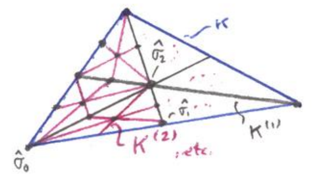
{vertices, those of τ and $\hat{\sigma}$

Notes: (i) $|K^{(1)}| = |K|$

(ii) Every simplex of $K^{(1)} \subset$ some simplex of K.

(iii) $K^{(1)} = \{(\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_n) : \sigma_0 < \sigma_1 < \dots < \sigma_n \in K\}$

(iv) $K^{(r)} := (K^{(r-1)})^{(1)}$



Definition: The mesh of a complex $K = \max\{\text{diam}(\text{star}(a,K)) : a \text{ a vertex of } K\}$.

Lemma 3.5: Given $\varepsilon > 0$, $\exists r$ such that mesh $K^{(r)} < \varepsilon$ (for given K).

Proof: Let l_K = maximum length of a 1-simplex in K. So mesh $K \le 2l_K$.

$l_{K^{(1)}} \le \frac{n}{n+1} l_K$, where $\dim K = n$ (since distance from vertex to barycentre $= \frac{n}{n+1}$(medium)).

So, $l_{K^{(r)}} \le (\frac{n}{n+1})^r l_K$, and $(\frac{n}{n+1})^r \to 0$ as $r \to \infty$

Theorem 3.6 (Simplicial Approximation Theorem): Let $\varphi : |K| \to |L|$ be continuous. For sufficiently large r, $\exists$ a simplicial approximation $f : |K^{(r)}| \to |L|$ to $\varphi : |K^{(r)}| \to |L|$

Proof: $\{\varphi^{-1}\text{star}(b,L) : b \text{ a vertex of } L\}$ is an open-cover of K-compact.

Lebesgue lemma $\Rightarrow$ for sufficiently large r, $\text{star}(a, K^{(r)}) \subset$ some $\varphi^{-1}(\text{star}(b,L))$ for each vertex $a \in K^{(r)}$. By lemma 3.4, f exists.

Corollary: (i) {Homotopy classes of maps $|K| \to |L|$} is countable: $\exists$ simplicial approximation in each class.

(ii) $\pi_1(|K|, a_0)$ is countable: by lemma 3.3 addendum - base point fixed.

(iii) $\pi_1(S^n) = \{1\}$ $\forall n \ge 2$: a simplicial approximation to a loop maps into 1-skeleton $\subsetneq S^n$.

Lemma 3.7: If $f, g : |K^{(r)}| \to |L|$ both approximate $\varphi : |K^{(r)}| \to |L|$, then $\forall \sigma \in K^{(r)}$, $\exists \tau \in L$ such that $f\sigma \le \tau$ and $g\sigma \le \tau$. (Say that f and g are contiguous).

Proof: If $x \in \mathring{\sigma}$, $\sigma \in K^{(r)}$, $\varphi(x) \in \mathring{\tau}$ some $\tau \in L$ (see lemma 3.3), $f(x) \in \tau$ and $g(x) \in \tau$. So, $f\sigma$ and $g\sigma$ are faces of τ.

Lemma 3.8: Given K and L, $\exists\ \delta > 0$ such that if ~~maps~~ maps $\varphi, \psi : |K| \to |L|$ are such that $d(\varphi, \psi) < \delta$, then $\exists$, for some r, a simplicial approximation to both φ and ψ, $f : |K^{(r)}| \to |L|$.

Proof: Choose δ so that diam $S \le 4\delta \Rightarrow S \subset \text{star}(b,L)$, some vertex $b \in L$. If $A \subset |L|$, let $A^* = \{x \in A : B(x, \varepsilon + \delta) \subset A \text{ for some } \varepsilon > 0\}$ - open. $\{(\text{star}(b,L))^* : b \text{ a vertex of } L\}$ is an open-cover of $|L|$, because $x \in |L| \Rightarrow B(x, 2\delta) \subset$ some star (b,L) $\therefore x \in (\text{star}(b,L))^*$, $\delta = \varepsilon$

↑diam = 4δ

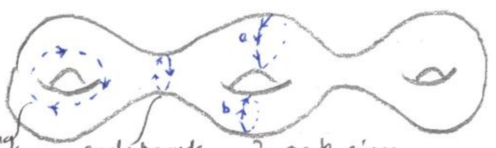
Select as before a large r and $f: |K^{(r)}| \to |L|$ such that if a is a vertex of K, $\varphi(\text{star}(a, K^{(r)})) \subset \text{star}(f(a), L)^* \subset \text{star}(f(a), L)$ $\therefore$ f is a simplicial approximation to φ.
$\psi(\text{star}(a, K^{(r)})) \subset \text{star}(f(a), L)$ since $d(\varphi, \psi) < \delta$ $\therefore$ f is a simplicial approximation to ψ.

4. Homology Groups of Complexes.

$$\text{Homology} = \frac{\text{Cycles}}{\text{Boundaries}} = \frac{\text{Something with no boundary}}{\text{Something that bounds}}$$

 - 1-simplex; endpoints are boundary. Join lots, eg:  boundaries cancel, so a cycle.

1-cycle bounding 2-dimensional object:  where  induces , so inner arrows cancel leaving cycle as boundary.



A simplex is ordered if we are given an ordering of its vertices. Write $\sigma = (a_0 \dots a_m)$ if $a_0 < \dots < a_m$.
Let K be a finite simplicial complex.

Definition: The nth chain group, $C_n(K)$, $n \geq 0$, is:

$$\frac{\text{free abelian group with generators (in 1-1 correspondance with) all } \sigma \in K \text{ with all possible orders, } \dim \sigma = n}{\langle (a_0 \dots a_n) - \varepsilon_\pi (a_{\pi_0} \dots a_{\pi_n}) : \pi \in \Sigma_n, \text{ symmetric group} \rangle}$$

ie, force relation $(a_0 \dots a_n) = \varepsilon_\pi (a_{\pi_0} \dots a_{\pi_n})$, so σ with orientation defined up to an even permutation is an oriented simplex.

By convention $C_n(K) = 0$ if $n < 0$ or $n > \dim K$.
An element of $C_n(K)$ is: $\sum_i \lambda_i \sigma_i$, $\lambda_i \in \mathbb{Z}$, σ_i an n-simplex with ordering.

Definition: The boundary homomorphism, $d_n: C_n(K) \to C_{n-1}(K)$ is the linear map defined on generators by: $d_n(a_0 \dots a_n) = \sum_{i=0}^n (-1)(a_0 \dots \overset{a_i}{\nearrow} \dots a_n)$, where $\nearrow$ indicates omit.

Example: 2-simplex: $d_n(a_0 a_1 a_2) = (a_1 a_2) - (a_0 a_2) + (a_1 a_2)$: "Flipping" $\to$ 

Lemma 4.1: d_n is well-defined.
Proof: $d(a_0 \dots a_{r-1} a_{r+1} a_r a_{r+2} \dots a_n) = \sum_{i=0}^{r-1} (-1)^i (a_0 \dots \overset{a_i}{\nearrow} \dots a_{r+1} a_r \dots a_n) + (-1)^r (a_0 \dots \overset{a_{r+1}}{\nearrow} \dots a_n)$
$+ (-1)^{r+1} (a_0 \dots \overset{a_r}{\nearrow} \dots a_n) + \sum_{i=r+2}^{n} (-1)^i (a_0 \dots a_{r+1} a_r a_{r+2} \dots \overset{a_i}{\nearrow} \dots a_n) = -d_n(a_0 \dots a_n)$, so well-defined.

Lemma 4.2: $d^2 = 0$, ie, $C_n(K) \xrightarrow{d_n} C_{n-1}(K) \xrightarrow{d_{n-1}} C_{n-2}(K)$ is zero.
Proof: $d^2(a_0 \dots a_n) = d \sum_{i=0}^n (-1)^i (a_0 \dots \overset{a_i}{\nearrow} \dots a_n) = \sum_i (-1)^i \left\{ \sum_{j=0}^{i-1} (-1)^j (a_0 \dots \overset{a_j}{\nearrow} \dots \overset{a_i}{\nearrow} \dots a_n) - \sum_{j=i+1}^{n} (-1)^j (a_0 \dots \overset{a_i}{\nearrow} \dots \overset{a_j}{\nearrow} \dots a_n) \right\} = 0$

$C_{n+1}(K) \xrightarrow{d} C_n(K) \xrightarrow{d} C_{n-1}(K)$.

$\underbrace{\mathrm{Ker}(d_n : C_n(K) \to C_{n-1}(K))}_{n\text{-cycles}} \supset \underbrace{\mathrm{Image}(d_{n+1} : C_{n+1}(K) \to C_n(K))}_{n\text{-boundaries.}}$

Definition: The n-boundaries, $B_n(K)$, is Image $d_{n+1} : C_{n+1}(K) \to C_n(K)$

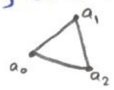
The n-cycles, $Z_n(K)$, is kernel $d_n : C_n(K) \to C_{n-1}(K)$.

$d^2 = 0 \Rightarrow B_n(K) \subset Z_n(K) \subset C_n(K)$

Definition: The n^{th} homology group, $H_n(K)$ of K is: $Z_n(K)/B_n(K)$

Example: $H_n(1 \text{ point}) = \begin{cases} \mathbb{Z}, & n=0 \\ 0, & n \neq 0, \end{cases} \quad \to \overset{C_1}{0} \to \overset{C_0}{\mathbb{Z}} \to \overset{C_{-1}}{0} \to$

Boundary of a 2-simplex:



$C_2 \xrightarrow{d} C_1 \longrightarrow C_0 \longrightarrow C_{-1}$

C_2: 0

C_1: $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$, free abelian, generators: $(a_0a_1), (a_1a_2), (a_2a_0)$

C_0: $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$, generators: a_0, a_1, a_2

C_{-1}: 0

$d(a_0a_1) = a_1 - a_0$, $d(a_1a_2) = a_2 - a_1$, $d(a_2a_0) = a_0 - a_2$. 1-cycles are: $n((a_0a_1), (a_1a_2), (a_2a_0))$.

$\therefore H_1(\Delta) = \mathbb{Z}$, $H_0(\Delta) = \mathbb{Z}$, $H_2(\Delta) = 0$

Note: H_0(path-connected simplex) $= \mathbb{Z}$, as $d(a_ia_j) = a_j - a_i$, so in $H_0(\cdots)$, $[a_j] = [a_i]$.

Suppose $f : |K| \to |L|$ is a simplicial map. Define $f_n : C_n(K) \to C_n(L)$ to be the linear extension of: $f(a_0 \cdots a_n) = \begin{cases} (fa_0, \ldots, fa_n) & \text{if } fa_i \text{ all distinct} \\ 0 & \text{otherwise.} \end{cases}$ – well-defined.

Lemma 4.3: (i) The following diagram commutes:

$$\begin{array}{ccccccc} \xrightarrow{d_{n+1}^K} & C_n(K) & \xrightarrow{d_n^K} & C_{n-1}(K) & \xrightarrow{d_{n-1}^K} \\ & \downarrow f_n & & \downarrow f_{n-1} & \\ \longrightarrow & C_n(L) & \xrightarrow{d_n^L} & C_{n-1}(L) & \xrightarrow{d_{n-1}^L} \end{array}$$

ie, $d_n^L f_n = f_{n-1} d_n^K \ \forall n$.

$\{f_n\}$ is a chain map between chain complexes.

(ii) If $f = 1_{|K|}$, then $f_n = 1_{C_n(K)}$

(iii) $(gf)_n = g_n f_n$.

Proof: If $fa_0, \ldots, fa_n$ are distinct, $d f_n(a_0 \cdots a_n) = \Sigma(-1)^i (fa_0 \cdots \widehat{fa_i} \cdots fa_n) = f_{n-1} d(a_0 \cdots a_n)$.

if $fa_r = fa_s$, $f_{n-1} d(a_0 \cdots a_n) = f_{n-1} d(a_0 \overset{\text{swap } a_r, a_s}{\cdots} a_n) = -f_{n-1} d(a_0 \cdots a_n) = 0$, which equals $d f_n(a_0 \cdots a_n)$

Corollary: f induces $\{f_n\}$ which induce $f_* : H_n(K) \to H_n(L)$ such that $1_* = 1$, $(fg)_* = f_* g_*$

Proof: By (i), $f_n Z_n(K) \subset Z_n(L)$, $f_n B_n(K) \subset B_n(L)$, and so induces $H_n(K) = Z_n(K)/B_n(K) \to Z_n(L)/B_n(L) = H_n(L)$

Later: Shall use the Simplicial Approximation Theorem to approximate $\varphi : |K| \to |L|$, continuous, by a simplicial map and use above. Will get: $\varphi_* : H_n(K) \to H_n(L)$ such that $1_* = 1$, $(\varphi\psi)_* = \varphi_* \psi_*$

Lemma 4.4: Suppose have chain complexes and chain maps $\{f_n\}, \{g_n\}$,

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(K) \arrow[r, "d"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] \arrow[dl] & C_n(K) \arrow[r, "d"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "h_n"] & C_{n-1}(K) \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "h_{n-1}"] & \cdots \arrow[dl] \\ \cdots \arrow[r] & C_{n+1}(L) \arrow[r, "d"'] & C_n(L) \arrow[r, "d"'] & C_{n-1}(L) \arrow[r] & \cdots \end{tikzcd}$$

and homomorphisms $h_n : C_n(K) \to C_{n+1}(L)$ $\forall n$ such that $f_n - g_n = d_{n+1}^L h_n + h_{n-1} d_n^K$, then $f_* = g_* : H_n(K) \to H_n(L)$ $\forall n$. [$\{h_n\}$ is a chain homotopy between chain maps, $\{f_n\}, \{g_n\}$]

Proof: If $x \in H_n(K)$, $x = [z]$, $z \in Z_n(K)$, $f_n(z) - g_n(z) = \underbrace{d h_n(z)}_{\text{boundary.}} + h_{n-1} \underbrace{d(z)}_{=0}$ $\therefore [f_n(z)] = [g_n(z)]$ in L.

Definition: The cone vK on complex K (v a vertex) is: $\{v\} \cup \{(a_0 \cdots a_r) \in K\} \cup \{(va_0 \cdots a_r) : (a_0 \cdots a_r) \in K\}$.

Lemma 4.5: $H_n(vK) = \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{otherwise,} \end{cases} \cong H_n(v)$.

Proof: $v \xrightarrow{i} vK \xrightarrow{j} v$. Simplicial maps, $j_* i_* = 1_* = 1$. Consider $ij : vK \to v \to vK$.

Define $h_n : C_n(vK) \to C_{n+1}(vK)$ by: $h_n(v\sigma) = 0$, $h_n(v) = 0$, $h_n(\sigma) = v\sigma$ $(\sigma \in K)$.

$(i_n j_n + dh_n + h_{n-1} d)(v\sigma) = 0 + 0 + v\sigma$.

$(i_n j_n + dh_n + h_{n-1} d)(v) = v + 0 + 0$

$(i_n j_n + dh_n + h_{n-1} d)(\sigma) = 0 + d(v\sigma) + v(d\sigma) = \sigma$, $\dim \sigma \geq 1$

$(i_n j_n + dh_n + h_{n-1} d)(\sigma) = v + d(v\sigma) + 0 = \sigma$, $\dim \sigma = 0$.

$\therefore h_n$ is a chain homotopy from 1_n to $i_n j_n$, so $i_* j_* = 1_* = 1$.

So i_*, j_* are isomorphisms.

Corollary: If σ is the complex of a simplex and its faces, $H_n(\sigma) = \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0. \end{cases}$

Corollary: Let σ be an $(n+1)$-simplex and all its faces. Let $\partial\sigma$ be the proper faces of σ. (So $\partial\sigma$ triangulates S^n). Then, $H_r(\partial\sigma) = \begin{cases} \mathbb{Z}, & r = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases}$

Proof: ($n=1$ done, assume $n \geq 2$). Have inclusion $i : \partial\sigma \hookrightarrow \sigma$. Have:

$$\begin{tikzcd}
 & 0 \arrow[r] & C_n(\partial\sigma) \arrow[r, "d"] \arrow[d, "i_n"] & C_{n-1}(\partial\sigma) \arrow[r, "d"] \arrow[d, "i_{n-1}"] & \cdots \arrow[r] & C_1(\partial\sigma) \arrow[r] \arrow[d, "i_1"] & C_0(\partial\sigma) \arrow[r] \arrow[d, "i_0"] & 0 \\
0 \arrow[r] & C_{n+1}(\sigma) \arrow[r] & C_n(\sigma) \arrow[r] & C_{n-1}(\sigma) \arrow[r] & \cdots \arrow[r] & C_1(\sigma) \arrow[r] & C_0(\sigma) \arrow[r] & 0
\end{tikzcd}$$

Via i_r, the two chain complexes are the same except in dimension $n+1$.

So, $H_r(\partial\sigma) = H_r(\sigma)$, $\forall r \leq n-1$, and $H_r(\sigma) = 0$ for $r \neq 0$.

$0 = H_{n+1}(\sigma) = Z_{n+1}(\sigma) / \cancel{B_{n+1}(\sigma) = 0}$. So $Z_{n+1}(\sigma) = 0$.

$\therefore \mathbb{Z} = C_{n+1}(\sigma) \xrightarrow{d_{n+1}} C_n(\sigma)$ is injective (since $\ker d_{n+1} = Z_{n+1} = 0$)

$\therefore B_n(\sigma) \cong \mathbb{Z}$, so $Z_n(\sigma) \cong \mathbb{Z}$, as $H_n(\sigma) = 0$.

$\therefore Z_n(\partial\sigma) \cong \mathbb{Z}$, and $B_n(\partial\sigma) = 0$, trivially, $\Rightarrow H_n(\partial\sigma) \cong \mathbb{Z}/0 \cong \mathbb{Z}$.

Definition: A sequence, $\cdots \to G_n \xrightarrow{f_n} G_{n-1} \xrightarrow{f_{n-1}} G_{n-2} \to \cdots$ of groups and homomorphisms is *exact* if $\ker f_{n-1} = \text{Image of } f_n$ $\forall n$ under consideration.

Examples: (i) $\cdots \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \cdots$ is exact iff all homology groups are zero.

(ii) If $0 \to G_2 \xrightarrow{f} G_1 \to 0$ is exact then f is an isomorphism.

(iii) $0 \to G \hookrightarrow H \xrightarrow{\text{projection}} H/G \to 0$ is exact when G is a normal subgroup of H.

If $0 \to A \xrightarrow{\alpha} B \to C \to 0$, then $C \cong B/\mathrm{Im}\,\alpha$ (first isomorphism theorem).

Theorem 4.6 (Mayer-Vietoris Theorem): Suppose that K is a simplicial complex and $K = L \cup M$, where L and M are subcomplexes ($\Rightarrow L \cap M$ a subcomplex). Then $\exists$ an exact sequence of homology groups: $\cdots \to H_n(L \cap M) \xrightarrow{\alpha_*} H_n(L) \oplus H_n(M) \xrightarrow{\beta_*} H_n(K) \xrightarrow{\Delta_n} H_{n-1}(L \cap M) \xrightarrow{\alpha_*} \cdots$

where, if inclusions are denoted,

$$\begin{tikzcd}
L \cap M \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"] & L \arrow[d, hook, "j_1"] \\
M \arrow[r, hook, "j_2"] & K
\end{tikzcd}$$

$\alpha_*(x) = ((i_1)_* x, (i_2)_* x)$,

$\beta_*(x, y) = (j_1)_* x - (j_2)_* y$.

Proof: $H_n(L) \oplus H_n(M)$ is the nth homology of chain complex $\{C_n(L) \oplus C_n(M), (d^L, d^M)\}$.

Definition of Δ_n: if $[z] \in H_n(K)$, means $z \in C_n(K)$, $dz = 0$.

β_n is surjective, so $\exists y \in C_n(L) \oplus C_n(M)$ with $\beta_n y = z$. So, since commuting, $\beta_{n-1} dy = d\beta_n y = 0$. Exactness $\Rightarrow \exists x \in C_{n-1}(L \cap M)$ with $\alpha_{n-1}(x) = dy$.

$$\begin{tikzcd}
0 \arrow[r] & C_n(L\cap M) \arrow[r, "\alpha_n"] \arrow[r, swap, "((i_1)_n,(i_2)_n)"] \arrow[d, "d"] & C_n(L)\oplus C_n(M) \arrow[r, "\beta_n"] \arrow[r, swap, "(j_1)_n-(j_2)_n"] \arrow[d, "d"] & C_n(K) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & C_{n-1}(L\cap M) \arrow[r, "\alpha_{n-1}"] \arrow[d, "d"] & C_{n-1}(L)\oplus C_{n-1}(M) \arrow[r, "\beta_{n-1}"] \arrow[d, "d"] & C_{n-1}(K) \arrow[r] \arrow[d, "d"] & 0 \\
& & & &
\end{tikzcd}$$

$$\begin{tikzcd}
& & \hat{y} \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\hat{x} \arrow[r, "\alpha_n"] & y-\bar{y}-d\hat{y} \arrow[d] & \bullet \arrow[r] & z-\bar{z} \\
& d(y-\bar{y}) & y \arrow[r] \arrow[d] & z \arrow[d] \\
x \arrow[r] \arrow[d] & dy \arrow[r] \arrow[d] & dy \arrow[r] & 0 \\
dx \arrow[r, "\alpha_{n-2}"] & \alpha_{n-2}dx & &
\end{tikzcd}$$

– exact (commutes)

$d\alpha_{n-1}(x) = d^2y = 0$, $\therefore \alpha_{n-2}\,dx = 0$. α_{n-2} is injective (exactness). $\therefore dx = 0$, so x is a cycle. Define $\Delta_n[z] = [x] \in H_{n-1}(L\cap M)$

Check Δ_n is well-defined.

Suppose $\bar{z}, \bar{y}, \bar{x}$ is another triple of choices, with $[\bar{z}] = [z]$. So, $z - \bar{z} \in B_n(K)$.
$\exists\ \hat{y} \in C_{n+1}(L)\oplus C_{n+1}(M)$ such that $d\beta_{n+1}\hat{y} = z - \bar{z}$, so $\beta_n d\hat{y} = z - \bar{z}$, and
$\beta_n(y-\bar{y}) = z - \bar{z}$, so $\beta_n(y - \bar{y} - d\hat{y}) = 0$.
$\therefore \exists\ \hat{x} \in C_n(L\cap M)$ with $\alpha_n\hat{x} = y - \bar{y} - d\hat{y}$. So, $\underset{\alpha_{n-1}(d\hat{x})}{\underset{\shortparallel}{d\alpha_n\hat{x}}} = \underset{\alpha_{n-1}(x-\bar{x})}{\underset{\shortparallel}{d(y-\bar{y})}} - \underbrace{d^2\hat{y}}_{=0}$

α_{n-1} injective $\Rightarrow d\hat{x} = x - \bar{x}$, so $[x] = [\bar{x}]$ $\therefore \Delta_n$ is well-defined.

<u>Note:</u> Δ_n is a homomorphism. If we have z_1, y_1, x_1 and z_2, y_2, x_2, then for $z_1 + z_2$ we can choose $y_1 + y_2$ and $x_1 + x_2$.

<u>Check exactness:</u>

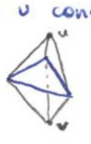
(a) At $H_n(L)\oplus H_n(M)$. $\beta_n\alpha_n = 0$, so $\beta_*\alpha_* = 0$. $\therefore \operatorname{Im}\alpha_* \subset \ker\beta_*$.
If $[k] \in \ker\beta_*$, $\beta_n k = dl$, some $l \in C_{n+1}(K)$
$= d\beta_{n+1} m$, some $m \in C_{n+1}(L)\oplus C_{n+1}(M)$, $= \beta_n dm = 0$.
So, $\beta_n(k - dm) = 0$. So $[k] = [k - dm] \in \operatorname{Im}\alpha_*$.

(b) At $H_n(K)$. If $[y] \in H_n(L)\oplus H_n(M)$, ie $dy = 0$, then $\beta_*[y] = [\beta_n y]$.
$\Delta_n[\beta_n y] = [x]$ such that $d_{n-1}x = dy = 0$ $\therefore x = 0$, ie $\Delta_n\beta_* = 0$, so $\operatorname{Im}\beta_n \subset \ker\Delta_n$.
Conversely, if $[z] \in \ker\Delta_n$, $\begin{matrix} \hat{x} & & y \to z \\ \downarrow & & \downarrow \\ x & \to & dy \end{matrix}$, $x = d\hat{x}$, some $\hat{x} \in C_n(L\cap M)$, as $[x] = 0$.

$d\alpha_n\hat{x} = dy$, $\therefore d(y - \alpha_n\hat{x}) = 0$, so $y - \alpha_n\hat{x}$ is a cycle.
$\beta_n(y - \alpha_n\hat{x}) = \beta_n y = z$. $[z] = \beta_*[y - \alpha_n\hat{x}]$, $\therefore [z] \in \operatorname{Im}\beta_*$.

(c) At $H_n(L\cap M)$. $\alpha_*\Delta = 0$, (obvious, as $\alpha_*\Delta_n[z] = [dy] = 0$), so $\operatorname{Im}\Delta \subset \ker\alpha_*$.
If $\alpha_*[x] = 0$, $dx = 0$. $\alpha_{n-1}x = dy$, some y. $d\beta_n y = \beta_{n-1}\alpha_{n-1}x = 0$.
$\therefore \beta_n y$ is a cycle, z, say, and $\Delta[z] = [x]$

<u>Example:</u> Let $\partial\sigma$ be the boundary of an n-simplex. Let K be $\underset{L}{\operatorname{cone}(u\,\partial\sigma)} \cup \underset{M}{\operatorname{cone}(v\,\partial\sigma)}$



$n \geq 2$, $H_r(L) = H_r(M) = \begin{cases} 0 & \text{if } r \neq 0 \\ \mathbb{Z} & \text{if } r = 0 \end{cases}$ (cone lemma)

Mayer-Vietoris: $\underset{0}{H_r(L)\oplus H_r(M)} \to H_r(K) \xrightarrow{\therefore\,\cong} H_{r-1}(L\cap M) \to \underset{0}{H_{r-1}(L)\oplus H_{r-1}(M)} \ldots$
$r \geq 2$,

$\therefore H_r(K) \cong H_{r-1}(L\cap M) = \begin{cases} \mathbb{Z} & \text{if } r-1 = n-1, \text{ or } 0 \\ 0 & \text{otherwise} \end{cases}$ $\therefore H_n(K) = \mathbb{Z}$, $H_r(K) = 0$, $2 \leq r \leq n-1$.

At end of sequence: $H_1(K) \xrightarrow{\therefore\ 0\text{-image}} H_0(L\cap M) \xrightarrow{0\text{-kernel}} H_0(L)\oplus H_0(M) \to H_0(K) \to 0$

$$\begin{array}{ccccc}
& \mathbb{Z} & \to & \mathbb{Z}\oplus\mathbb{Z} & \to \mathbb{Z} \\
& 1 & \xrightarrow{\text{injection}} & (1,\ 1) & \\
& \shortparallel & & (1,\ 0) & \longmapsto 1 \\
& [\text{vertex}] & & (0,\ 1) & \longmapsto -1
\end{array}$$

[$L\cap M$, L, M connected]

$\therefore H_1(K) = 0$.

Example: If $K = L \vee M$ (ie, $K = L \cup M$, and $L \cap M = \{\text{one point}\}$), $H_r(K) \cong H_r(L) \oplus H_r(M)$, $r > 0$

5. Invariance of Homology Groups.

Lemma 5.1: Let $\alpha : \{\text{vertices of } K^{(1)}\} \to \{\text{vertices of } K\}$ be such that $\forall \sigma \in K$, $\alpha\hat{\sigma}$ is a vertex of σ. Then α extends to a simplicial map $\alpha : K^{(1)} \to K$ which is an approximation to $1 : |K^{(1)}| \to |K|$.

Proof: If $\tau \in K^{(1)}$, $\mathring{\tau} \subset \mathring{\sigma}$, some unique $\sigma \in K$. α(vertices of τ) are vertices of σ.
$\therefore$ $\alpha\{\text{vertices of } \tau\}$ are vertices of some face of σ, so α extends to a simplicial map.
If v is a vertex of τ, $\mathring{\tau} \subset \mathring{\sigma} \subset \text{star}(\alpha v, K)$, with αv some vertex of σ. True $\forall$ simplexes τ with v as a vertex, so $1.\text{star}(v, K^{(1)}) \subset \text{star}(\alpha v, K)$, ie, α is an approximation to 1.

Note: if σ is an n-simplex of K, $\sigma = \alpha\tau$ for just one n-simplex τ, $\tau \in K^{(1)}$, and $\mathring{\tau} < \mathring{\sigma}$.
(Use induction on n).

Lemma 5.2: $\exists$ chain map $\theta_n : C_n(K) \to C_n(K^{(1)})$ $\forall n$, $(d\theta_n = \theta_{n-1} d)$, such that if σ is an n-simplex of K, $\theta_n \sigma = \sum_{\mathring{\tau} \subset \mathring{\sigma}} \pm \tau$ (some choice with $\pm$).

Proof: Let θ_0 be inclusion. Suppose have θ_r defined for $r < n$, and that $\dim\sigma = n$.
Define $\theta_n \sigma = \hat{\sigma}(\theta_{n-1}(d\sigma))$, where $\hat{\sigma}(\sum \lambda_i (a_0^i \ldots a_{n-1}^i)) = \sum \lambda_i (\hat{\sigma} a_0^i \ldots a_{n-1}^i)$.
θ_n is well-defined. Inductively, $d\theta_{n-1} = \theta_{n-2} d$.
$d\theta_n \sigma = d(\hat{\sigma}(\theta_{n-1} d\sigma)) = \theta_{n-1} d\sigma - \underbrace{\hat{\sigma}(d(\theta_{n-1} d\sigma))}_{= \hat{\sigma}\theta_{n-2} dd\sigma = 0}$, so $d\theta_n = \theta_{n-1} d$.

Note: if $\dim\sigma = n$, $\alpha_n \theta_n \sigma = \pm\sigma$ (using note to lemma 5.1).
$\therefore$ $\alpha_n \theta_n$ is an isomorphism $C_n(K) \to C_n(K)$ (self-inverse).
$\therefore$ $\alpha_* \theta_*$ is an isomorphism $H_n(K) \to H_n(K)$

Lemma 5.3: Suppose $\{f_n\}, \{g_n\} : C_n(K) \to C_n(L)$ are chain maps such that:
(i) f_0 and g_0 map vertices to vertices,
(ii) For each n-simplex $\sigma \in K$, $\exists$ a cone $\Lambda_\sigma \subset L$ (subcomplex) such that $f_n \sigma$ and $g_n \sigma \in C_n(\Lambda_\sigma) \subset C_n(L)$
(iii) $\tau < \sigma \Rightarrow \Lambda_\tau \subset \Lambda_\sigma$.
Then, $f_* = g_* : H_r(K) \to H_r(L)$ $\forall r$.

Proof: (Recall lemma 4.4). Construct chain homotopy $h_n : C_n(K) \to C_{n+1}(L)$.
If $\dim\sigma = 0$, $f_0\sigma$ and $g_0\sigma$ are vertices in $C_0(\Lambda_\sigma)$ ← path-connected.
$\exists$ 1-chain, call it $h_0\sigma \in C_1(\Lambda_\sigma)$ such that $f_0\sigma - g_0\sigma = d(h_0\sigma)$.
Suppose for $r < n$ and $\dim\sigma = r$, have $h_r\sigma \in C_{r+1}(\Lambda_\sigma)$ such that $f_r - g_r = dh_r + h_{r-1}d$.
Suppose $\dim\sigma = n > 0$. $d(f_n - g_n - h_{n-1}d)\sigma = (f_{n-1} - g_{n-1} - dh_{n-1})d\sigma = (h_{n-2}d)d\sigma = 0$.
But $(f_n - g_n - h_{n-1}d)\sigma \in C_n(\Lambda_\sigma)$ by properties (ii) and (iii).
So, $(f_n - g_n - h_{n-1}d)\sigma$ is a cycle. $H_n(\Lambda_\sigma) = 0$ (it is a cone - see lemma 4.5)
$\therefore$ $(f_n - g_n - h_{n-1}d)\sigma = d(\text{some chain}) = d(h_n\sigma)$, say. This defines h_n.
$\therefore$ $f_* = g_*$ by lemma 4.4.

Corollary: $\alpha_*: H_n(K^{(1)}) \to H_n(K)$ and $\theta_*: H_n(K) \to H_n(K^{(1)})$ are mutually inverse isomorphisms.

Proof: Have that $\alpha_*\theta_*$ is an isomorphism. Consider 1 and $\theta_n\alpha_n: C_n(K^{(1)}) \to C_n(K^{(1)})$

If $\tau \in K^{(1)}$, $\mathring{\tau} \subset \mathring{\sigma}$, $\sigma \in K$, $\alpha\tau$ is a face of σ. Take $\Lambda_\tau = \sigma^{(1)} = \hat{\sigma}(d\sigma)^{(1)}$ - cone.

By lemma 5.3, $\theta_*\alpha_* = 1_* = 1$. $\therefore$ θ_* and α_* are both isomorphisms, and $\theta_* = \alpha_*^{-1}$.

$$\begin{tikzcd} H_n(K) \arrow[r, shift left, "\theta_*"] \arrow[rrrr, bend left, "\text{Definition: } \theta_*^{(r)}"] & H_n(K^{(1)}) \arrow[l, shift left, "\alpha_*"] \arrow[r, shift left, "\theta_*"] & H_n(K^{(2)}) \arrow[l, shift left, "\alpha_*"] \arrow[r, shift left, "\theta_*"] & \cdots \arrow[l, shift left, "\alpha_*"] \arrow[r, shift left, "\theta_*"] & H_n(K^{(r)}) \arrow[l, shift left, "\alpha_*"] \arrow[llll, bend left, "\text{Definition: } \alpha_*^{(r)}"] \end{tikzcd}$$

Theorem 5.4: For every continuous map $\varphi: |K| \to |L|$ there is, for each n, a well-defined homomorphism $\varphi_*: H_n(K) \to H_n(L)$ such that $1_* = 1$ and $(\psi\varphi)_* = \psi_*\varphi_*$

Proof: Let $f: |K^{(r)}| \to |L|$ be a simplicial approximation to φ (Theorem 3.6).

Define $\varphi_* = f_*\theta_*^{(r)}$. Suppose $\bar{f}: |K^{(r)}| \to |L|$ is another simplicial approximation (same r).

$\bar{f}$ and f are contiguous (lemma 3.7), ie $\bar{f}\sigma$ and $f\sigma$ are both faces of some $\tau \in L$.

τ is a cone (simplex). Let Λ_σ be minimal such τ. By lemma 5.3, $\bar{f}_* = f_*$.

$\therefore$ $\bar{f}_*\theta_* = f_*\theta_*$.

Now suppose $\hat{f}: |K^{(s)}| \to |L|$ is a simplicial approximation, with $s<r$, wlog.

α is an approximation to 1, $\therefore$ $\hat{f}\alpha^{(r-s)}: |K^{(r)}| \to |L|$ is an approximation to φ.

By above, $\hat{f}_*\alpha_*^{(r-s)} = f_*$. $f_*\theta_*^{(r)} = \hat{f}_*\alpha_*^{(r-s)}\theta_*^{(r)} = \hat{f}_*\theta_*^{(s)}$, so φ is well-defined.

$1_* = 1$ is trivial. Suppose we have continuous maps $|K| \xrightarrow{\varphi} |L| \xrightarrow{\psi} |M|$ and simplicial approximations $|K^{(s)}| \xrightarrow{f} |L^{(r)}| \xrightarrow{g} |M|$. gf is a simplicial approximation to $\psi\varphi$.

$(\psi\varphi)_* = (gf)_*\theta_*^{(s)} = g_*f_*\theta_*^{(s)} = \underbrace{g_*\theta_*^{(r)}}_{\psi_*}\underbrace{\alpha_*^{(r)}f_*\theta_*^{(s)}}_{(\text{approximation to } \varphi)_*} = \psi_*\varphi_*$

Corollary: If $\varphi: |K| \to |L|$ is a homeomorphism, then φ_* is an isomorphism.

Proof: $\varphi\varphi^{-1} = \varphi^{-1}\varphi = 1$, $\therefore$ $(\varphi\varphi^{-1})_* = (\varphi^{-1}\varphi)_* = 1_*$, $\therefore$ $\varphi_*(\varphi^{-1})_* = (\varphi^{-1})_*\varphi_* = 1$.

So φ_* and φ_*^{-1} are mutually inverse isomorphisms.

If X is a space homeomorphic to $|K|$, some complex K, define $H_r(X) \cong H_r(K)$.

Ie, $H_r(X)$ defined up to isomorphism.

Example: $H_r(S^n) = \begin{cases} \mathbb{Z} & \text{if } r=0 \text{ or } n \\ 0 & \text{otherwise} \end{cases}$, so $S^n \cong |\partial(\sigma^{n+1})|$

Theorem 5.5: If $\varphi \underset{F}{\simeq} \psi: |K| \to |L|$, then $\varphi_* = \psi_*$.

Proof: $F: |K| \times [0,1] \to |L|$. Let $F_t(x) = F(x,t)$, so $F_t: |K| \to |L|$, with $F_0 = \varphi$, $F_1 = \psi$.

(Lemma 3.8 $\Rightarrow$ $\exists\, \delta > 0$ such that two maps $|K| \to |L|$ within δ have the same simplicial approximation)

F is continuous and $|K| \times [0,1]$ is compact, so F is uniformly continuous. So $\exists\, n > 0$ such that $d(F_t, F_{t+1/n}) < \delta$. Then, $\varphi_* = (F_0)_* \underbrace{=}_{\text{have same approximation}} (F_{0+1/n})_* \underbrace{=}_{\text{similarly, but a different one}} (F_{0+2/n})_* = \cdots = (F_1)_* = \psi_*$.

Corollary: If φ is a homotopy equivalence, φ_* is an isomorphism.

Proof: $|K| \underset{\chi}{\overset{\varphi}{\rightleftarrows}} |L|$. So, $\chi\varphi \simeq 1$, $\varphi\chi \simeq 1$. So, $(\chi\varphi)_* = 1_*$, $(\varphi\chi)_* = 1_*$, so $\chi_*\varphi_* = 1$, $\varphi_*\chi_* = 1$.

Example:  $S^1 \xrightarrow[\simeq]{\varphi}$ Annulus. $H_1(\text{Annulus}) = \mathbb{Z}$.

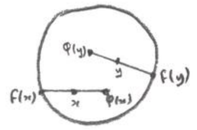
Circle has generators:  - sum of simplexes, $\therefore$ so does annulus.

Applications: (i) $S^n \not\simeq S^m$, $S^n \not\cong S^m$ $(n \neq m)$. . $H_n(S^n) \cong \mathbb{Z}$ $(m, n \geq 1)$
$H_n(S^m) = 0$.

(ii) $\mathbb{R}^n \not\cong \mathbb{R}^m$ - remove point: $\mathbb{R}^n - \{0\} \simeq S^{n-1}$.

Theorem 5.6 (Brouwer Fixed Point Theorem): Any continuous $\varphi: B^n \to B^n$ has a fixed point.

Proof: Suppose $\varphi(x) \neq x \ \forall x$. Define $f: B^n \to S^{n-1}$ thus:



(Take $n \geq 2$)

$$\begin{tikzcd} & B^n \arrow[dr, "f"] & \\ S^{n-1} \arrow[ur, "i"] \arrow[rr, "1"'] & & S^{n-1} \end{tikzcd}$$

So,

$$\begin{tikzcd} & \underset{0}{H_{n-1}(B^n)} \arrow[dr, "f_*"] & \\ \underset{\mathbb{Z}}{H_{n-1}(S^{n-1})} \arrow[ur, "i_*"] \arrow[rr, "1"'] & & \underset{\mathbb{Z}}{H_{n-1}(S^{n-1})} \end{tikzcd}$$

- So, $f_* i_* = 1 : \mathbb{Z} \to \mathbb{Z}$, and $i_* = 0$ - contradiction.

6. Surfaces.

Calculation of homology of (certain) surfaces.

Example (i): Let L be:  - g pairs of bands "interlocking", added to the boundary of a disc.

One single closed curve = boundary of L.

L deformation retracts to $S^1 \vee S^1 \vee \ldots \vee S^1$ ($2g$ times)

$H_r(L) \cong H_r(S^1 \vee \ldots \vee S^1)$. $H_0(L) = \mathbb{Z}$
$H_1(L) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ ($2g$ copies)
$H_r(L) = 0$, $r \geq 2$

Let M be a disc:  Identify boundary of M with boundary of L.
$L \cup M$ is a surface with no boundary.

Mayer-Vietoris sequence:

$$\underset{0}{H_2(L \cap M)} \longrightarrow \underset{0}{H_2(L)} \oplus \underset{0}{H_2(M)} \longrightarrow H_2(L \cup M) \longrightarrow$$

$$\underset{\mathbb{Z}}{H_1(L \cap M)} \xrightarrow{?=0} \underset{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}{H_1(L)} \oplus \underset{0}{H_1(M)} \xrightarrow{\therefore \text{ onto}} H_1(L \cup M) \xrightarrow{\therefore \text{ zero} \Rightarrow}$$

$$\underset{\mathbb{Z}}{H_0(L \cap M)} \xrightarrow{\text{injective} \Rightarrow} \underset{\mathbb{Z}}{H_0(L)} \oplus \underset{\mathbb{Z}}{H_0(M)} \longrightarrow \underset{\mathbb{Z}}{H_0(L \cup M)} \longrightarrow 0$$

$1 \longmapsto (1, 1)$

$$H_1(L\cap M) \xrightarrow{i_{1*},\, i_{2*}} H_1(L) \oplus H_1(M)$$
$$1 \longmapsto (?\,,\,0)$$
Σ(1-simplexes of $L\cap M$) $\to a_1-a_1, b_1-b_1, a_2-a_2, \dots = 0$ $\therefore ? = 0$ (see: -shrinking to blue line - red lines cancel)

$\therefore$ Exactness $\Rightarrow H_1(L)\oplus H_1(M) \xrightarrow{\text{isomorphism}} H_1(L\cup M)$ $\therefore H_1(L\cup M) \cong \mathbb{Z}\oplus\cdots\oplus\mathbb{Z}$ (2g copies)

Exactness $\Rightarrow H_2(L\cup M) \xrightarrow{\text{isomorphism}} H_1(L\cap M)$, $\therefore H_2(L\cup M)\cong\mathbb{Z}$

$$\therefore H_r(L\cup M) = \begin{cases} \mathbb{Z}, & r=0 \\ \mathbb{Z}\oplus\cdots\oplus\mathbb{Z} \text{ (2g copies)}, & r=1 \\ \mathbb{Z}, & r=2 \\ 0, & \text{otherwise}\end{cases}$$

So surface(g_1) is not homeomorphic to surface(g_2).

Example (ii): Same L, but M a Möbius band: ◎-M. $M \simeq S^1$

$L\cap M \cong S^1$. $H_r(M) = H_r(S^1)$

$$H_1(L\cap M) \xrightarrow{i_{1*},\, i_{2*}} H_1(L)\oplus H_1(M)$$
$$\mathbb{Z} \qquad \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{2g} \oplus \mathbb{Z}$$
$$1 \xmapsto{\text{injection}} (0\,,\,2)$$

(See: -shrinking Möbius band to circle: two 1-simplexes gives $1\mapsto 2$)

$\therefore H_2(L\cup M) = 0$

$$H_1(L\cup M) = \frac{H_1(L)\oplus H_1(M)}{\text{Image } H_1(L\cap M)} \cong \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{2g \text{ copies}} \oplus \mathbb{Z}/2\mathbb{Z}.$$

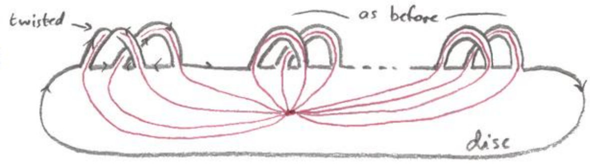
$H_0(L\cup M) = \mathbb{Z}$.

Exercise: ◎ $\cup$ ◎ (2 Möbius bands) = Klein bottle, K.

common boundary

$H_1(K) = \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$

Example (iii): Let L be:



L deformation retracts to $S^1\vee\cdots\vee S^1$ (2g times).

M still a disc - identify boundaries.

$$H_1(L\cap M) \longrightarrow H_1(L)\oplus H_1(M).$$
$$\mathbb{Z} \longrightarrow \mathbb{Z}\oplus\cdots\oplus\mathbb{Z}\oplus 0$$
$$1 \longmapsto a_1-a_1, b_1+b_1, a_2-a_2, b_2-b_2, \dots$$ - so injection. So $H_2(L\cup M) = 0$.

$H_1(L\cup M) = \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}$ ($2g-1$ $\mathbb{Z}$'s).

⊛ Classifying Surfaces.

Definition: An *n-manifold without boundary* is a space (Hausdorff) M such that $x\in M \Rightarrow \exists$ open U with $x\in U\subset M$ and $U\cong\mathbb{R}^n$.

Examples: $\mathbb{R}^n$, S^n, $\mathbb{RP}^n$, $S^{n-r}\times S^r$, T^2 (torus), K (Klein bottle), $L_{p,q}$.

If $n=1$, and M is compact, connected, then $M\cong S^1$.

If $n=2$, we call M a *surface*.

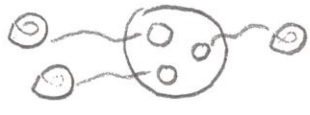
If M_1, M_2 are connected n-manifolds, form their connected sum, $M_1 \# M_2$:

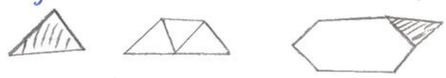
(i) remove interior of an n-ball from M_1, M_2. For example: 

(ii) Identify resulting boundary. So: 

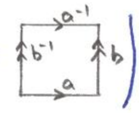
Quote: Any compact 2-manifold M is triangulable, $M \cong |K|$
(Assume no boundary). Each 1-simplex of K is a face of two 2-simplexes.

Theorem 6.1: Let M be a compact connected 2-manifold without boundary. Then M is homeomorphic to one and only one of:
M_g $(g = 0, 1, 2, \ldots)$, $M_0 = S^2$, $M_g = T^2 \# \cdots \# T^2$ (g times), where T^2 = torus, $S^1 \times S^1$.
N_h $(h = 1, 2, \ldots)$; $N_h = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ (h times), where $\mathbb{RP}^2 = S^2/x \sim \pm x$ = 

[M_g:  N_h:  N_2 = Klein bottle.
$M_g \# N_h = N_{2g+h}$]

Proof: M a surface, $M \cong |K|$. K a 2-complex - each edge is a face of two 2-simplexes.
Regard 2-simplexes as disjoint and reassemble 

Obtain: M is homeomorphic to an n-gon with its edges to be identified in pairs.
Label edges to be identified: $a, a^{\pm1}, b, b^{\pm1}, c, c^{\pm1}, \ldots$

-ve pair: a, a^{-1}, if anticlockwise around n-gon, induces opposite orientation on edge. (For example: Torus: 

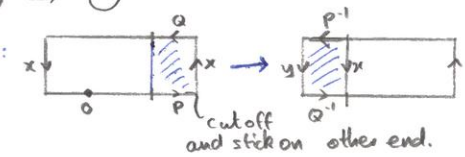
)

+ve pair: a, a, if anticlockwise around n-gon, induces same orientation on edge.

Labels around n-gon give a word: Γ. ($aba^{-1}b^{-1}$ in torus example)

Rules: (i) Abolish -ve edges besides their mates: 

(ii) $\cdots P x Q \cdots x \cdots \sim \cdots y \cdots P^{-1} y Q^{-1} \cdots$
symbol
subwords
same 2-manifold

(See: 

)

(iii) $\cdots P x Q \cdots x^{-1} \cdots \sim \cdots y \cdots Q y^{-1} P \cdots$

Proceed: (a) Move +ve pairs in Γ to the front. $\Gamma = A\overset{P}{B} x C x D \sim A y \overset{Q}{C B^{-1}} y D$ (rule (iii)) $\sim A z z B C^{-1} D$.
Eventually, $\Gamma \sim AR$, $A = x_1 x_1 x_2 x_2 \cdots x_n x_n$, R is -ve pairs.

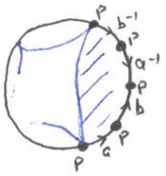
(b) Apply rule (i) as much as possible. Then $\exists$ a pair of interlocking -ve pairs in R (or $R = \phi$), ie $\cdots x \cdots y \cdots x^{-1} \cdots y^{-1} \cdots$.
Move to front: $\Gamma = ABx C y D x^{-1} E y^{-1} F$ (rule (iii)) $\sim A z C y D z^{-1} \overset{Q}{BE} y^{-1} F$ (rule (iii) backwards)
$\sim A z C t BED z^{-1} t^{-1} F \sim A u BEDC t u^{-1} t^{-1} F \sim A u v u^{-1} v^{-1} BEDCF$.

Eventually, $\Gamma \sim AB$, where A is +ve pairs, B is interlocking -ve pairs, ie $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots$

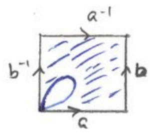
(c) One positive pair converts all pairs to be positive, ie $xxaba^{-1}b^{-1} \sim yyddcc$ (exercise).

So, $\Gamma \sim$ (i) ϕ.
(ii) $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$.
(iii) $x_1 x_1 x_2 x_2 \cdots x_n x_n$.

With (ii):

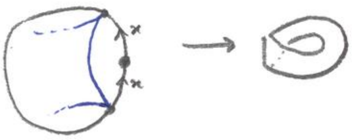


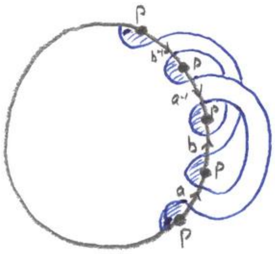
Identifying "nose-to-tail" means all P's are the same point. $\Rightarrow$



$\Rightarrow T^2 \# \cdots \# T^2$ (g copies)

With (iii):

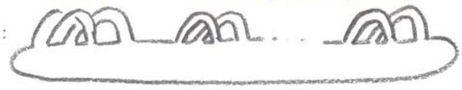




$M_g = L +$ disc (L from before)

$H_1(M_g) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (2g copies)

Repeat for N_h: h odd: above L + Möbius band

h even:  $\cup$ disc

$H_1 = \mathbb{Z}/2\mathbb{Z} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{h-1}$