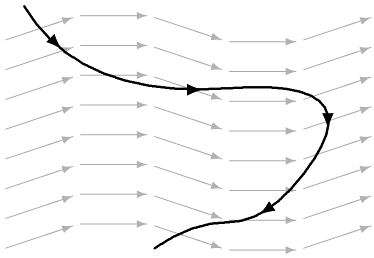
Some stuff about integrals and differentials

Line Integrals

Suppose that we have a small bead on a wire. If a constant force pushes the bead along the wire, then the work done by the force is given by the formula 'force $\times$ distance'. If the force varies as the bead travels, then we calculate the work done as the 'integral of force d(distance)', i.e. $\int f \, dx$.

But suppose that the wire is lying in a region with some flow swirling about. In other words, the bead is being affected by a vector field. We'll push the bead along the wire, and investigate the effect of the flow on the bead as it moves.



As the bead travels through the flow on its wire, the force exerted upon it by the flow doesn't just depend on the strength of the flow, but also on its direction: a tailwind will help push the bead on, but a headwind will provide some resistance. What about a crosswind? This won't really affect us at all – the bead is on its wire and can't be deflected sideways, and the crosswind is neither going to push us onward or hold us back.

At a given point $\mathbf{x}$, the flow has a vector value $\mathbf{F}(\mathbf{x})$, and the bead is moving in a direction given by a small displacement $\delta\mathbf{x}$. We care about how much of the flow is in the direction we're going, which we calculate using the scalar product: $\mathbf{F}(\mathbf{x}) \cdot \delta\mathbf{x}$. This gives us the work done by the force on the bead as it moves through that displacement.

We sum up these small contributions as $W = \sum \mathbf{F}(\mathbf{x}) \cdot \delta\mathbf{x}$.

How do we actually calculate such a sum? As in the scalar case, we parametrise the curve in order to 'straighten it out' to an integral over an interval.

Let $\gamma : [a, b] \to \mathbb{R}^3$ describe the path. That is, $\gamma(t) = \mathbf{x}(t) = (x(t), y(t), z(t))$. We obtain

$$\begin{aligned} W &= \sum \mathbf{F}(\gamma(t)) \cdot \delta\mathbf{x}(t) \\ &= \sum (F_1(\gamma(t)), F_2(\gamma(t)), F_3(\gamma(t))) \cdot \left(\frac{\delta x}{\delta t}, \frac{\delta y}{\delta t}, \frac{\delta z}{\delta t} \right) \delta t \\ &= \sum \left(F_1 \frac{\delta x}{\delta t} + F_2 \frac{\delta y}{\delta t} + F_3 \frac{\delta z}{\delta t} \right) \delta t \end{aligned}$$

where we have just written F_i for $F_i(\gamma(t))$, with the change of variables being understood.

In the limit, we obtain an integral

$$W = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$$

we can calculate in the usual way.

Now, let's return to the above expression $W = \sum \mathbf{F}(\mathbf{x}) \cdot \delta\mathbf{x}$. We wrote this before we made the parametrisation of the path, and we could write this same expression for whatever path we used – it was the introduction of γ which turned this general notation into the actual calculation we were able to do. In other words, this expression is 'waiting to be summed', and when we tell it the path we wish to use, it becomes a sum.

Accordingly, let us write $\displaystyle\int \mathbf{F} \cdot d\mathbf{x}$ or $\displaystyle\int F_1 dx + F_2 dy + F_3 dz$ as notation for the limit version, and think of it as 'something that is waiting to be integrated'.

We can't actually calculate this yet, because we haven't told it the path. We can think of it as saying to us:

There's a flow $\mathbf{F}$ *around. Its effect in the* x *direction is* F_1*, in the* y *direction is* F_2*, and in the* z *direction is* F_3*. Where would you like to go?*

We give it our path γ, and it says:

Ah, you're travelling that *much in the* x *direction, are you? Then you don't get all of* F_1*, just a contribution which is proportional to how* x*-wards you're going. Okay, you can have* $\frac{dx}{dt}$ *of* F_1*. And similarly for the* y *and* z *directions. So your overall contribution from* $\mathbf{F}$ *when going that way is* $\frac{dx}{dt}$ *of* F_1*,* $\frac{dy}{dt}$ *of* F_2*, and* $\frac{dz}{dt}$ *of* F_3*.*

We get the integral for W that we had before, and we integrate it along γ.

Conservative fields

Suppose that our vector field is the gradient of a scalar field, say $\mathbf{F} = \boldsymbol{\nabla} f$, for some $f : \mathbb{R}^3 \to \mathbb{R}$.

In Cartesian coordinates, we have $\mathbf{F} = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}\right)$.

Choosing our path γ, our formula for W then becomes

$$W = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt$$

where $f = f(x(t), y(t), z(t))$. By the chain rule, this says

$$W = \int_a^b \frac{df}{dt}\, dt = \Big[f(\gamma(t))\Big]_a^b = f(\gamma(b)) - f(\gamma(a))$$

In other words, the work done doesn't depend on the path γ, just on the endpoints.

A field in which such integrals don't depend on the path is called *conservative*. For if the work done in moving between any two points doesn't depend upon the path, then the work done in moving around any closed curve is 0 – we have conserved energy.

Is the reverse true? That is, if $\mathbf{F}$ is a field for which all line integrals around closed curves give 0, must $\mathbf{F}$ be the gradient of some scalar field? The answer is yes: we pick a base point $\mathbf{x}_0$, and define

$$f(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{x}$$

along any path. This is well-defined (i.e., independent of the path we use) since the field is conservative, and it can be shown that $\boldsymbol{\nabla} f = \mathbf{F}$.

Exact differentials

In our formula for a line integral, we have the expression $\mathbf{F} \cdot d\mathbf{x} = F_1 dx + F_2 dy + F_3 dz$. This is called a *differential*. There is a whole theory of differentials, but for now we can think of them, as earlier, as things that are 'waiting to become an integral'.

Or, as something that's waiting for a parametrisation of x, y, z in terms of t, so that it can become $\left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$.

Suppose we have $f : \mathbb{R}^3 \to \mathbb{R}$, and we parametrise x, y, z in terms of t. Then the chain rule gives

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Writing

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz$$

gives one of the differentials that's waiting to become an integrand. A differential of this form is called an *exact* differential, because it is precisely 'd of a function f'.

More generally, a differential of the form $\mathbf{F} \cdot d\mathbf{x} = F_1 dx + F_2 dy + F_3 dz$ is exact if there exists some scalar function f such that this differential equals df as written above.

The discussion earlier tells us that $\mathbf{F} \cdot d\mathbf{x}$ is exact if and only if $\mathbf{F}$ is conservative.

Testing for exactness

If $f : \mathbb{R}^3 \to \mathbb{R}$ is a sufficiently well-behaved function (meaning suitable derivatives are continuous – any function we'll meet in the Vector Calculus course will be well-behaved), then we have:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Now, suppose we have field $\mathbf{F} = (F_1, F_2, F_3)$ and we think it might equal $\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.

We investigate $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_2}{\partial x}$. If our suspicions are correct, then these should both equal $\frac{\partial^2 f}{\partial x \partial y}$.

In particular, they should be equal. Similarly, we should find $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$ and $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$.

If any of these tests fail, then $\mathbf{F}$ is not the gradient of a scalar.

(If all three of these tests pass, must $\mathbf{F}$ be a gradient? That's a question for next time.)