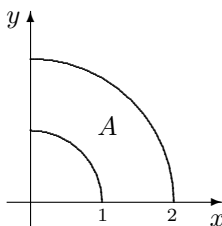


Jacobians

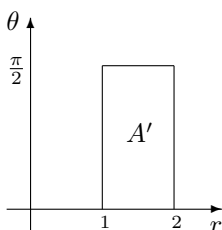
Suppose we wish to integrate $\int_A \cos(x^2 + y^2) \, dA$, where A is the region between two quarter-circles:



We decide to change to polar coordinates, by setting $x = r \cos \theta$, $y = r \sin \theta$.

The integrand transforms from $\cos(x^2 + y^2)$ to $\cos r^2$.

The region above satisfies $r \in [1, 2]$ and $\theta \in [0, \frac{\pi}{2}]$, which gives a much nicer shape if we draw axes labelled with r and θ , as we see:



So the limits of the integral will now be much easier. But we must also remember to introduce the Jacobian, which is the determinant of a matrix of partial derivatives:

$$J = \begin{vmatrix} \partial x / \partial r & \partial x / \partial \theta \\ \partial y / \partial r & \partial y / \partial \theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r$$

We then declare that $dx \, dy = |J| \, dr \, d\theta = r \, dr \, d\theta$, and obtain

$$\int_A \cos(x^2 + y^2) \, dA = \int_0^{\pi/2} \int_1^2 (\cos r^2) r \, dr \, d\theta$$

which we can now do more easily.

So, what was going on? What did the change of variables really do, and what did the Jacobian do?

It is common to think of the change of variables procedure as ‘transforming from x, y into r, θ ’, since we were initially in the x, y world, and we ended up in the r, θ world. However, it is better to think of this transformation as going the other way – or rather, ‘having come from the other way’.

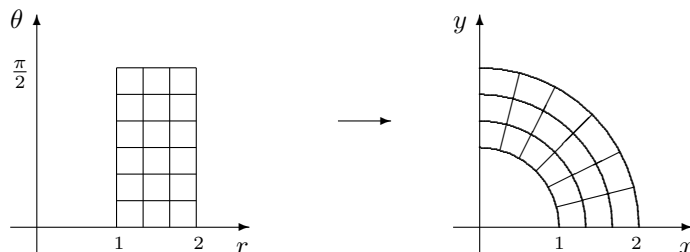
While we initially had expressions in x, y , we then wrote ‘ $x = r \cos \theta$, $y = r \sin \theta$ ’, so we were declaring x, y to be *functions of* r, θ . In other words, in some sense, we were assuming that there were some r, θ already lying around, and we were defining x, y in terms of them.

With this view in mind, we could think of the change of variables as ‘pulling back’ from the x, y world into the r, θ world.

Let us now think about how the integration process changes. The integral can be approximated by subdividing the region of integration into little boxes and estimating the value of the function in that region. We multiply this estimate by the area of the little box to find the contribution to the integral, and then we add up all of the contributions.

When we change variables, the estimate of the value of the function is easy: in the above example we just turned $\cos(x^2 + y^2)$ into $\cos(r^2)$. However, what happens to the little box? We need to know how the area of this changes in order to make the above approximation. This is the role of the Jacobian.

We have the following (with the ‘pulling back’ idea meaning that we’ll draw the r, θ picture first):



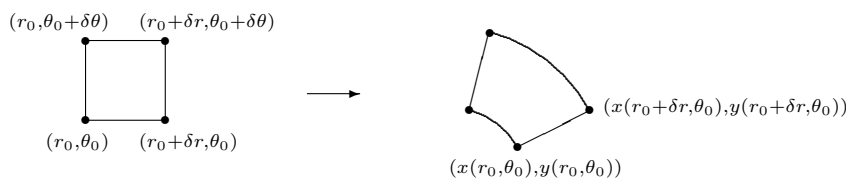
In the r, θ picture, the little boxes have nice dimensions, say they’re $\delta r \times \delta \theta$. Over in the x, y picture, these boxes have distorted. Notice, however, that these ‘boxes’ all have one dimension the same: their ‘depth’ (i.e., as measured outwards from the origin) is constant, and is still δr . But their ‘width’ increases as we move outwards – it is proportional to the distance we move away from the origin, and so turns out to be $r \delta \theta$. So these new ‘boxes’ have area roughly equal to $r \delta r \delta \theta$ (if we pretend they’re rectangles, that is).

We obtain a fairly plausible justification of the Jacobian rule that the x, y world dA equals $r dr d\theta$.

Can we prove this more rigorously?

Take one small box in the r, θ world, of sides δr and $\delta \theta$, with bottom-left corner at (r_0, θ_0) .

We change variables, defining $x = x(r, \theta)$ and $y = y(r, \theta)$. (We know what the actual functions are in our example, of course, but let’s be more general.)



We approximate the sides of this new ‘box’. Consider the right-most point. Its x -coordinate is:

$$x(r_0 + \delta r, \theta_0) = x(r_0, \theta_0) + \frac{\partial x}{\partial r} \delta r + \dots$$

where we have used a Taylor series expansion in the first component.

Similarly, its y coordinate is

$$y(r_0 + \delta r, \theta_0) = y(r_0, \theta_0) + \frac{\partial y}{\partial r} \delta r + \dots$$

Hence the bottom-right edge can be approximated as the vector

$$(x(r_0 + \delta r, \theta_0), y(r_0 + \delta r, \theta_0)) - (x(r_0, \theta_0), y(r_0, \theta_0)) \approx \left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r} \right) \delta r$$

Next, consider the left-most point. Its x -coordinate is:

$$x(r_0, \theta_0 + \delta \theta) = x(r_0, \theta_0) + \frac{\partial x}{\partial \theta} \delta \theta + \dots$$

where we have used a Taylor series expansion in the second component.

Similarly, its y coordinate is

$$y(r_0, \theta_0 + \delta\theta) = y(r_0, \theta_0) + \frac{\partial y}{\partial \theta} \delta\theta + \dots$$

Hence the bottom-left edge can be approximated as the vector

$$(x(r_0, \theta_0 + \delta\theta), y(r_0, \theta_0 + \delta\theta)) - (x(r_0, \theta_0), y(r_0, \theta_0)) \approx \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta} \right) \delta\theta$$

Now that we have estimates for these two vectors, we can approximate the area of the new box. Treating it as roughly a parallelogram (it doesn't look like it in the picture, but for very small δr and $\delta\theta$ it will be a better match), we can use the 'two-dimensional vector product' to find the area. Namely:

$$\delta A \approx \begin{vmatrix} \partial x / \partial r & \partial x / \partial \theta \\ \partial y / \partial r & \partial y / \partial \theta \end{vmatrix} \delta r \delta \theta$$

Hence, taking the limit, we obtain $dA = |J| dr d\theta$, with this determinant above being precisely the Jacobian.

(Why do we take the modulus of the determinant, and use $|J|$ rather than J ? We want to estimate the *area* of the little boxes. One way to guarantee this is to use $|J|$ for the scale factor, and to promise that we always integrate from 'low to high', so that we really are doing $\delta r \times \delta\theta$, and not accidentally $-\delta r \times \delta\theta$.)

And finally, we see how the change of variables takes effect. Let us return to our example. To picture what happens, let's imagine the process in terms of approximations and sums. (Although, of course, when integrating we don't really go via sums.)

In the x, y world, we have $\int_A \cos(x^2 + y^2) dA$.

We approximate this as a sum, $\sum_{A\text{-boxes}} \cos(x^2 + y^2) \delta A$.

We 'pull back' into the r, θ world, giving $\sum_{A'\text{-boxes}} (\cos r^2) |J| \delta r \delta \theta$.

And we return to an integral, $\int_{A'} (\cos r^2) r dr d\theta$.

The same process works in three dimensions, with little change. The transformation now turns little cubes into distorted little 'cubes', whose sides we estimate as above, and whose volume we estimate with a 3×3 determinant.

Exercise

After the first pictures on page 2, there was an informal justification for why the plane polar Jacobian was r , by observing that the sides of the new boxes were δr and $r \delta\theta$. It should be clear from a similar picture that in three-dimensional cylindrical coordinates, the new boxes have sides δr , $r \delta\theta$ and δz , confirming that the Jacobian there is also r .

Convince yourself that the spherical polar Jacobian is indeed $r^2 \sin \theta$, by reasoning what the sides of the new boxes are – i.e., what multiples of δr , $\delta\theta$, $\delta\phi$.

Please let me know of any corrections: glt1000@cam.ac.uk