Green's theorem in the plane

Green's theorem in the plane. For functions $P(x, y)$ and $Q(x, y)$ defined in $\mathbb{R}^2$, we have

$$\oint_C (P\,dx + Q\,dy) = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where C is a simple closed curve bounding the region A.

Vector Calculus is a "methods" course, in which we apply these results, not prove them. Here is a sketch proof.

We'll show that $\iint_A \frac{\partial P}{\partial y}\,dx\,dy = -\oint_C P\,dx$, and $\iint_A \frac{\partial Q}{\partial x}\,dx\,dy = \oint_C Q\,dy$.

Assume that R has "nice y-ranges", in the sense that for each fixed x, the range of y-integration is an interval, say $[y_-(x), y_+(x)]$. Let the range of x-integration be $[x_-, x_+]$. Then

$$\begin{aligned}
\iint_A \frac{\partial P}{\partial y}\,dx\,dy &= \int_{x_-}^{x_+} \left(\int_{y_-(x)}^{y_+(x)} \frac{\partial P}{\partial y}\,dy\right) dx \\
&= \int_{x_-}^{x_+} P(x, y_+(x)) - P(x, y_-(x))\,dx \\
&= -\int_{x_-}^{x_+} P(x, y_-(x))\,dx - \int_{x_+}^{x_-} P(x, y_+(x))\,dx \\
&= -\int_{C_b} P(x, y)\,dx - \int_{C_t} P(x, y)\,dx \\
&= -\oint_C P(x, y)\,dx
\end{aligned}$$

where C_t, C_b are the top and bottom parts of C. (Remember that C is traversed anticlockwise.)

Similarly, assume R has "nice x-ranges", i.e., that for each fixed y, the range of x-integration is an interval, say $[x_-(y), x_+(y)]$. Let the range of y-integration be $[y_-, y_+]$.

$$\begin{aligned}
\iint_A \frac{\partial Q}{\partial x}\,dx\,dy &= \int_{y_-}^{y_+} \left(\int_{x_-(y)}^{x_+(y)} \frac{\partial Q}{\partial x}\,dx\right) dy \\
&= \int_{y_-}^{y_+} Q(x_+(y), y) - Q(x_-(y), y)\,dy \\
&= \int_{y_-}^{y_+} Q(x_+(y), y)\,dy + \int_{y_+}^{y_-} Q(x_-(y), y)\,dy \\
&= \int_{C_l} Q(x, y)\,dy + \int_{C_r} Q(x, y)\,dy \\
&= \oint_C Q(x, y)\,dy
\end{aligned}$$

where C_l, C_r are the left and right parts of C.

Combining these two calculations gives the required answer for A.

Any fairly nice A can be subdivided into such nice regions. When summing the path subintegrals, all internal edges cancel out, leaving the integral around the outer boundary C.