Arc-length, curvature, torsion, etc.

Suppose that I go for a drive around town, trying to decide which is the scariest corner. If my speed isn't constant then I might find it hard to tell. For example, if I compare a shallow bend driven at 60mph to a sharp bend driven at 10mph, then I might end up thinking that the shallow bend was the scary one. The higher speed has given me a false impression of how tight the bend really was.

In order to compare corners properly, I'll drive at a constant speed. So now, if a corner *feels* scarier, that's because it *is* scarier.

You also decide to drive at a constant speed to compare the corners, so each of us can decide which are the scary corners. But can we compare? If I drive everywhere at 60mph and you drive at 10mph, then we'll both conclude that a particular corner is the scariest, but I'll think it's 'very scary' and you'll think it's 'not too bad really'.

In order for our results to be comparable, we'll both drive at the *same* constant speed. So now, if I try out the corners in my town while you try out the corners in yours, we can still come up with a decent comparison.

In fact, we'll drive everywhere at speed 1.

Let's now consider the curve $\mathbf{x}(t) = (4t, 3\cos t, 3\sin t)$ in $\mathbb{R}^3$. (What does this look like?)

This vector $\mathbf{x}(t)$ is the position vector, telling us where we'll be at time t.

The velocity vector is then $\mathbf{x}'(t) = \dfrac{d\mathbf{x}}{dt} = (4, -3\sin t, 3\cos t)$.

Hence the current speed is $|\mathbf{x}'(t)| = \left(16 + 9\sin^2 t + 9\cos^2 t\right)^{1/2} = (16+9)^{1/2} = 5$. This is too fast!

Writing s for 'displacement' (as in the familiar $s = ut + \frac{1}{2}at^2$ formula), we have $\frac{ds}{dt} = 5$. So we have $s = 5t$. This makes sense: driving at speed 5 for time t takes us distance $5t$.

To obtain speed 1, we invert this: $t = s/5$, and we consider $\mathbf{x}(s) = \big(4(s/5), 3\cos(s/5), 3\sin(s/5)\big)$.

The parameter is now the displacement variable s. Originally, when in terms of t, we were saying 'after time t, where on the curve am I?', which clearly depends upon speed; but now we're saying 'once I have travelled distance s, where on the curve am I?', which is independent of any speed worries (although it's clearly equivalent to travelling at speed 1). This is a good parametrisation, called the **arc-length parametrisation**.

Let's just verify that it has speed 1. We have $\mathbf{x}'(s) = \frac{d\mathbf{x}}{ds} = \big(\frac{4}{5}, -\frac{3}{5}\sin(s/5), \frac{3}{5}\cos(s/5)\big)$, and then $|\mathbf{x}'(s)| = 1$.

From now on, we'll just write things like $\mathbf{x}'$ rather than $\mathbf{x}'(s)$, with 'with respect to s' or 'evaluated at s' being understood. And while we're here, let's define it properly: $\mathbf{x}'$ is called the **tangent**, rather than the 'velocity', and is written $\mathbf{t}$.

Now that we have this nice description of the curve, we can investigate its corners. Since we are travelling at speed 1, there is never any acceleration in the direction of motion, and any acceleration we feel must be perpendicular to the direction of motion, caused entirely by the curve itself. The larger the magnitude of this acceleration is, the tighter the curve.

Compare swinging a ball on a string. The acceleration our curve is feeling above is the same as that along the string due to the centripetal force, towards the centre of the rotation. Indeed, as our curve travels, we could think of it at each point as describing a tiny arc segment of some circle. (But, unless the curve is actually drawing a circle, this visualisation needs continual adjustment – the centre and radius of the circle to which the tiny arc segment belongs will change.)

The acceleration vector is $\mathbf{x}''$, and we'll define the scalar function κ and unit vector $\mathbf{n}$ by $\mathbf{x}'' = \kappa\mathbf{n}$. We call κ the **curvature**, and $\mathbf{n}$ the **principal normal**. The vector $\mathbf{n}$ is precisely 'along the string', and κ is 'how tight is the corner'.

Note that the shorter the imagined string, the tighter the bend, and hence the larger the curvature. Indeed, the radius R of the instantaneous small circle is $1/\kappa$, and we call R the **radius of curvature**.

Note also that the algebra confirms our visualisable definition that $\mathbf{t}$ and $\mathbf{n}$ are perpendicular: since $|\mathbf{t}| = 1$, we have $\mathbf{t} \cdot \mathbf{t} = 1$, and differentiating gives $2\mathbf{t} \cdot \mathbf{t}' = 0$. So, unless $\kappa = 0$ (in which case there isn't a well-defined principal normal), we have $\mathbf{t} \cdot \mathbf{n} = 0$.

In the example above, we have $\mathbf{x}'' = \big(0, -\frac{3}{25}\cos(s/5), -\frac{3}{25}\sin(s/5)\big)$, and hence $\kappa = 3/25$ and $\mathbf{n} = (0, -\cos(s/5), -\sin(s/5))$.

If we draw the curve $\mathbf{x}$ in this example, we'd find that it's a circular helix, wrapping around a cylinder lying along the x-axis. It looks a bit like the edge of a corkscrew. But this leads us to the next question: can we tell whether it's twisting clockwise or anticlockwise? Every corkscrew I've ever encountered needs a clockwise twist to get it into the cork (although I'd like to find an anticlockwise one – partly because mathematicians generally prefer anticlockwise, but mostly to confuse my friends).

Consider first the curve drawn by $(0, 3\cos t, 3\sin t)$. This is just a circle in the yz-plane, about the x-axis. Imagine drawing it repeatedly (i.e., just keep increasing t, rather than stopping at 2π), and imagine that the x coordinate was then gradually increased. The curve would start moving out away from the yz-plane, orbiting the x-axis, describing a corkscrew shape. This is what our main example is doing, since its x-coordinate is given by $4t$.

How do we describe this algebraically? We wish to see if the curve is moving away from that instantaneous circle we mentioned earlier. To do this, we will investigate the 'twist' of the curve. But how do we detect twisting? The tangent vector isn't very helpful, since it always points in the direction of travel, and if we twist the curve then the tangent vector doesn't notice. But the principal normal sticks out away from the tangent vector, and we can test if it is rotating about the tangent.

To this end, we introduce the **binormal** vector, defined as $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. Being a vector product, this is perpendicular to both the tangent and the normal, and it 'sticks up' from the curve. (To return to our driving analogy: the tangent is the direction we're driving in, the normal is the dirction given by pointing out of the car window, and the binormal is sort of like a rear spoiler, only pointing straight up from the car.)

With the circular curve $(0, 3\cos t, 3\sin t)$ above, we have $\mathbf{t} = (0, -3\sin t, 3\cos t)$, and so $\mathbf{n} = (0, -\cos t, -\sin t)$, pointing directly at the origin. Then the binormal is $\mathbf{b} = \mathbf{t} \times \mathbf{n} = (3, 0, 0)$, pointing out along the x-axis. If we then gradually increase the x-coordinate as before, then the normal starts to lift out of the yz-plane, pointing slightly more along the x-axis.

This is how we measure the twist: does the normal move towards or away from the binormal? Recall that given a vector $\mathbf{v}$ and a unit vector $\mathbf{u}$, the component of $\mathbf{v}$ in the direction $\mathbf{u}$ is given by $\mathbf{v} \cdot \mathbf{u}$. Here, we wish to know whether the derivative $\mathbf{n}'$ is pointing more towards or away from $\mathbf{b}$, and **torsion** satisfies $\tau = \mathbf{n}' \cdot \mathbf{b}$.

(This is not the usual definition of torsion, but I think that this is a nicely visualisable consequence of the definition. The usual definition is via $\mathbf{b}' = -\tau\mathbf{n}$, and the consequence above comes out through the orthonormality of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$.)

In the circle example, we find $\tau = 0$, which agrees with the curve never leaving the yz-plane.

In the original corkscrew example, we have

$$\begin{aligned}\mathbf{n}' &= \left(0, \tfrac{1}{5}\sin(s/5), -\tfrac{1}{5}\cos(s/5)\right)\\ \mathbf{b} &= \left(\tfrac{4}{5}, -\tfrac{3}{5}\sin(s/5), \tfrac{3}{5}\cos(s/5)\right) \times \left(0, -\cos(s/5), -\sin(s/5)\right) \;=\; \left(\tfrac{3}{5}, \tfrac{4}{5}\sin(s/5), -\tfrac{4}{5}\cos(s/5)\right)\end{aligned}$$

and hence $\tau = \mathbf{n}' \cdot \mathbf{b} = 4/25$.

So the corkscrew has positive torsion. This makes sense, if we think about what we generally like the 'positive sense' of things to be: we like anticlockwise for rotations about axes, and we like upwards. If we place the world with the yz-plane horizontal and the positive x-axis pointing upwards, then the curve is travelling anticlockwise about the x-axis, and upwards.

If we draw the same curve in the reverse direction, then it still has positive torsion – it's the same curve after all. (To push the the corkscrew into the other end of the cork, we still twist our hand the same way.) But now it's travelling clockwise about the x-axis, and downwards. Both 'positive senses' have been reversed. But if we took a reflection of the curve (or a left-handed corkscrew), then only one of the senses has been reversed, and such a curve would have negative torsion (it would 'twist down into the plane' rather than 'lifting up out of it').

Did we need to find $t = s/5$?

One last thing. At the start of our example, we found $\frac{ds}{dt} = |\mathbf{x}'(t)| = 5$. We then integrated to obtain $s = 5t$, and inverted this to find $t = s/5$. We then used this parametrisation throughout.

But what would we do if $\frac{ds}{dt}$ wasn't such a simple function?

Even for a simple ellipse $\mathbf{x}(t) = (a\cos t, b\sin t)$ in the plane, we have $\frac{ds}{dt} = \sqrt{a^2\cos^2 t + b^2\sin^2 t}$.

This would not be fun to integrate, and even if we could do it the resulting integral would not be easy to invert. However, there is such an inverse function, even if we can't write it down.

So, since it's the inverse to $s(t)$, let's just write it as $t(s)$.

We would then write the ellipse as $\mathbf{x}(s) = (a\cos t(s), b\sin t(s))$.

Then $\mathbf{t} = \dfrac{d\mathbf{x}}{ds} = \dfrac{d\mathbf{x}}{dt}\,\dfrac{dt}{ds} = \dfrac{d\mathbf{x}}{dt}\Big/\dfrac{ds}{dt}$.

So we don't actually need to know the inverse function $t(s)$, and can rely on knowing just $\frac{ds}{dt}$ throughout.