

Vector Calculus: Example Sheet 1

Part IA, Lent Term 2026

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Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Vector Differential Operators

1. Obtain the equation of the plane that is tangent to the surface $z = 3x^2y \sin(\frac{1}{2}\pi x)$ at the point $x = y = 1$.

Take East to be in the direction $(1, 0, 0)$ and North to be $(0, 1, 0)$. In which direction will a marble roll if placed on the surface at $x = 1, y = \frac{1}{2}$?

2. (i) Let $\phi(\mathbf{x})$ be a scalar field and $\mathbf{v}(\mathbf{x})$ a vector field. Show, using suffix notation, that

$$\nabla \cdot (\phi \mathbf{v}) = \nabla \phi \cdot \mathbf{v} + \phi \nabla \cdot \mathbf{v}, \quad \nabla \times (\phi \mathbf{v}) = \nabla \phi \times \mathbf{v} + \phi \nabla \times \mathbf{v}.$$

- (ii) Evaluate the divergence and curl of each of the following:

$$r\mathbf{x}, \quad (\mathbf{a} \cdot \mathbf{x})\mathbf{b}, \quad \mathbf{a} \times \mathbf{x}, \quad \frac{\mathbf{x} - \mathbf{a}}{|\mathbf{x} - \mathbf{a}|^3},$$

where $r = |\mathbf{x}|$ and \mathbf{a}, \mathbf{b} are constant vectors.

- (iii) The vector field $\mathbf{B}(\mathbf{x})$ is everywhere parallel to the normals of a family of surfaces $f(\mathbf{x}) = \text{constant}$. Show that $\mathbf{B} \cdot (\nabla \times \mathbf{B}) = 0$.

- (iv) The vector fields \mathbf{F} and \mathbf{G} are everywhere parallel, non-zero and solenoidal. Show that $\mathbf{F} \cdot \nabla(F/G) = 0$, where $F = |\mathbf{F}|$ and $G = |\mathbf{G}|$.

3. Use suffix notation to show that for vector fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$,

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v}$$

and

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla(\frac{1}{2}|\mathbf{u}|^2) - \mathbf{u} \times (\nabla \times \mathbf{u}).$$

4. Verify directly that the vector field

$$\mathbf{u}(\mathbf{x}) = (e^x(x \cos y + \cos y - y \sin y), e^x(-x \sin y - \sin y - y \cos y), 0)$$

is irrotational and express it as the gradient of a scalar field ϕ . Check also that \mathbf{u} is solenoidal and show that it can be written as the curl of a vector field $\mathbf{v} = (0, 0, \psi)$ for some function ψ .

- * 5. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a homogeneous function of degree n , i.e., $f(k\mathbf{x}) = k^n f(\mathbf{x})$ for all $k \in \mathbb{R}$. By differentiating with respect to k , or otherwise, show that $\mathbf{x} \cdot \nabla f = nf$.

- * 6. Suppose that $\mathbf{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a solenoidal vector field. Show that $\mathbf{F} = \nabla \times \mathbf{A}$ where

$$\mathbf{A}(\mathbf{x}) = \int_0^1 \mathbf{F}(t\mathbf{x}) \times (t\mathbf{x}) dt.$$

This is an example of a *homotopy formula*, in this case for finding a vector potential for a given solenoidal field. What goes wrong if the domain of definition of \mathbf{F} in \mathbb{R}^3 contains a hole of some kind?

Orthogonal Curvilinear Coordinates

7. The vector field $\mathbf{A}(\mathbf{x})$ is, in Cartesian, cylindrical and spherical polar coordinates respectively,

$$\mathbf{A}(\mathbf{x}) = -\frac{1}{2}y\mathbf{e}_x + \frac{1}{2}x\mathbf{e}_y = \frac{1}{2}\rho\mathbf{e}_\phi = \frac{1}{2}r \sin \theta \mathbf{e}_\phi$$

(where \mathbf{e}_ϕ has two different meanings). Compute $\nabla \times \mathbf{A}$ in each coordinate system and check that your answers agree.

8. If \mathbf{a} is a constant vector and $r = |\mathbf{x}|$, verify that $\nabla r^n = nr^{n-2}\mathbf{x}$ using (i) Cartesian coordinates and suffix notation, (ii) Taylor's theorem in the form $df = \nabla f \cdot d\mathbf{x}$, (iii) cylindrical polar coordinates, (iv) spherical polar coordinates. Verify also that $\nabla(\mathbf{a} \cdot \mathbf{x}) = \mathbf{a}$ using methods (i) and (ii).

* Furthermore, verify that $\nabla(\mathbf{a} \cdot \mathbf{x}) = \mathbf{a}$ using methods (iii) and (iv). [Hint: you will need to be careful about the components of \mathbf{a} with respect to the relevant bases, which are not constant.]

9. (i) Use the Chain Rule to express $\partial f / \partial x$ in terms of $\partial f / \partial \rho$ and $\partial f / \partial \phi$ for a function $f(\rho, \phi, z)$ with respect to cylindrical polar coordinates. Combine this with similar calculations and with expressions for the basis vectors \mathbf{e}_ρ , \mathbf{e}_ϕ and \mathbf{e}_z to show that

$$\nabla f = \frac{\partial f}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{\partial f}{\partial z} \mathbf{e}_z.$$

(ii) We deduce from this result that

$$\nabla = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z}$$

and we also know that $\partial \mathbf{e}_\rho / \partial \phi = \mathbf{e}_\phi$, $\partial \mathbf{e}_\phi / \partial \phi = -\mathbf{e}_\rho$, while all other derivatives of the basis vectors are zero. Derive an expression for $\nabla \cdot \mathbf{A}$ where $\mathbf{A}(\mathbf{x})$ is an arbitrary vector field given in cylindrical polars by $\mathbf{A} = A_\rho \mathbf{e}_\rho + A_\phi \mathbf{e}_\phi + A_z \mathbf{e}_z$.

* Also derive expressions for $\nabla \times \mathbf{A}$ and for the Laplacian $\nabla^2 f$ of a scalar function $f(\rho, \phi, z)$.

Differential Geometry of Curves

10. Sketch the *astroid* curve in the plane given parametrically by $\mathbf{x}(t) = (a \cos^3 t, a \sin^3 t)$, $0 \leq t \leq 2\pi$. Calculate $\dot{\mathbf{x}}(t)$ at each point and hence find the curve's total length.

11. A circular helix is given by

$$\mathbf{x}(t) = (a \cos t, a \sin t, ct), \quad t \in \mathbb{R}$$

where a, c are constants. Calculate the tangent \mathbf{t} , curvature κ , principal normal \mathbf{n} and binormal \mathbf{b} , and show that the torsion is $\tau = c/(a^2 + c^2)$. Give a sketch of the curve indicating the directions of the vectors $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$.

12. Show that a planar curve $\mathbf{x}(t) = (x(t), y(t), 0)$ has curvature

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

Use this result to find the minimum and maximum values of the curvature on the ellipse $x^2/a^2 + y^2/b^2 = 1$, and comment on the case $a = b$.

* 13. The tangent vector at each point on a curve is parallel to a non-vanishing vector field $\mathbf{H}(\mathbf{x})$. Show that the curvature of the curve is given by $\kappa = |\mathbf{H} \times (\mathbf{H} \cdot \nabla)\mathbf{H}| / |\mathbf{H}|^3$.