# Vector Calculus: Example Sheet 2

### Part IA, Lent Term 2025



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*Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.*

## Line Integrals

**1.** Evaluate explicitly each of the line integrals

$$\int (x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z), \qquad \int (y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}z), \qquad \int (y\,\mathrm{d}x - x\,\mathrm{d}y + e^{x+y}\,\mathrm{d}z),$$

along (i) the straight line path from the origin to $(1, 1, 1)$, and (ii) the parabolic path given parametrically by $(x, y, z) = (t, t, t^2)$ from $t = 0$ to $t = 1$.

For which of these integrals do the two paths give the same results, and why?

**2.** Consider the two vector fields $\mathbf{F}(\mathbf{x}) = (3x^2yz^2, 2x^3yz, x^3z^2)$ and $\mathbf{G}(\mathbf{x}) = (3x^2y^2z, 2x^3yz, x^3y^2)$. Compute the line integrals $\int \mathbf{F}\,.\,\mathrm{d}\mathbf{x}$ and $\int \mathbf{G}\,.\,\mathrm{d}\mathbf{x}$ along the following paths, each of which consist of straight line segments joining the specified points:

(i) $(0, 0, 0) \to (1, 1, 1)$,
(ii) $(0, 0, 0) \to (0, 0, 1) \to (0, 1, 1) \to (1, 1, 1)$,
(iii) $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1)$.

Are either of the differentials $\mathbf{F}\,.\,\mathrm{d}\mathbf{x}$ or $\mathbf{G}\,.\,\mathrm{d}\mathbf{x}$ exact?

**3.** A curve $C$ is given parametrically by

$$\mathbf{x}(t) = \big(\cos(\sin nt)\cos t, \cos(\sin nt)\sin t, \sin(\sin nt)\big), \quad 0 \leqslant t \leqslant 2\pi,$$

where $n$ is some fixed integer. Using spherical polar coordinates, or otherwise, sketch the curve. Show that if

$$\mathbf{F}(\mathbf{x}) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

and $C$ is traversed in the direction of increasing $t$ then $\oint_C \mathbf{F}\,.\,\mathrm{d}\mathbf{x} = 2\pi$.

Find a scalar function $f$ such that $\mathbf{F}(\mathbf{x}) = \nabla f$. Comment on this, given your prior result.

## Area Integrals

**4.** Use the substitution $x = r\cos\phi$, $y = \frac{1}{2}r\sin\phi$ to show that

$$\int_D \frac{x^2}{x^2+4y^2}\,\mathrm{d}A = \tfrac{3}{4}\pi$$

where $D$ is the region between the two ellipses $x^2 + 4y^2 = 1$, $x^2 + 4y^2 = 4$.

**5.** A closed curve $\mathscr{C}$ in the $z = 0$ plane consists of the arc of the parabola $y^2 = 4ax$ $(a > 0)$ between the points $(a, \pm 2a)$, and the straight line joining $(a, \mp 2a)$. The area enclosed by $\mathscr{C}$ is $\mathscr{D}$. Show, by calculating the integrals explicitly, that

$$\int_{\mathscr{C}} (x^2y\,\mathrm{d}x + xy^2\,\mathrm{d}y) = \int_{\mathscr{D}} (y^2 - x^2)\,\mathrm{d}A = \frac{104}{105}a^4.$$

6. The region $D$ is bounded by the segments $x = 0, 0 \leqslant y \leqslant 1$; $y = 0, 0 \leqslant x \leqslant 1$; $y = 1, 0 \leqslant x \leqslant \frac{3}{4}$; and an arc of the parabola $y^2 = 4(1-x)$. Consider a mapping into the $(x, y)$-plane from the $(u, v)$-plane defined by the transformation $x = u^2 - v^2$, $y = 2uv$. Sketch $D$ and also the two regions in the $(u, v)$-plane that are mapped into it. Hence evaluate

$$\int_D \frac{\mathrm{d}A}{(x^2+y^2)^{1/2}}.$$

* 7. Let $f(x)$ be a smooth function such that $\int_1^\infty \frac{1}{x} f(x)\,\mathrm{d}x$ exists. By considering an appropriate double integral involving $f'$ and changing the order of integration, show that

$$\int_0^\infty \frac{f(bx)-f(ax)}{x}\,\mathrm{d}x = f(0)\log(a/b), \quad a, b > 0.$$

### Volume Integrals

8. Calculate the volume within the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ by using a suitable change of variables.

9. (i) Compute the volume of a cone of height $h$ and radius $a$ using cylindrical polars.
   (ii) The apex of the cone is placed at the origin and its axis aligned with the positive $z$-axis so that its base lies in the plane $z = h$. The density of the cone is then given by $\rho_0 \cos\theta$ in *spherical* polars. Show that its total mass is $\frac{2}{3}\pi\rho_0 h^3\big(\sqrt{1+a^2/h^2}-1\big)$.

10. A tetrahedron $V$ of uniform density $\rho_0$ has vertices at $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and the origin. Find its centre of mass, defined by $\mathbf{X} = M^{-1}\int_V \rho_0 \mathbf{x}\,\mathrm{d}^3\mathbf{x}$ where $M$ is the total mass. [*Answer:* $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.]

11. Let $(u, v, w)$ be a set of orthogonal curvilinear coordinates for $\mathbb{R}^3$. Show that a volume element is given by

$$\mathrm{d}V = h_u h_v h_w\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

Confirm that in cylindrical and spherical polars, $\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z$ and $\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$ respectively. Draw diagrams to illustrate these two results.

* 12. Let $a, b, c > 0$. Using the new variables $u = x/y$, $v = xy$, $w = yz$, or otherwise, show that

$$\int_{x=0}^\infty \int_{y=0}^1 \int_{z=0}^x x e^{-ax/y - bxy - cyz}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2a(a+b)(a+b+c)}.$$

* 13. Compute the volume of the region $V$ defined by the intersection of three cylinders as follows:

$$V = \big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant 1, y^2 + z^2 \leqslant 1, z^2 + x^2 \leqslant 1\big\}.$$

[*This question merely demonstrates, for fun, how difficult volume integrals can be. Answer:* $8\big(2-\sqrt{2}\big)$.]

### Green's Theorem in the Plane

14. Consider the line integral

$$I = \oint_C (-x^2 y\,\mathrm{d}x + xy^2\,\mathrm{d}y)$$

where $C$ is a closed curve traversed anticlockwise in the $(x, y)$-plane.

   (i) Evaluate $I$ when $C$ is a circle of radius $R$ centred at the origin. Use Green's theorem to relate the results for $R = b$ and $R = a$ to an area integral over an appropriate region, and calculate the area integral directly.
   (ii) Now suppose that $C$ is the boundary of a square centred at the origin with sides of length $l$. Show that $I$ is independent of the orientation of the square in the $(x, y)$-plane.