Vector Calculus: Example Sheet 1

Part IA, Lent Term 2025

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Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Vector Differential Operators

1. Obtain the equation of the plane that is tangent to the surface $z = 3x^2 y \sin(\frac{1}{2}\pi x)$ at the point $x = y = 1$.

Take East to be in the direction $(1, 0, 0)$ and North to be $(0, 1, 0)$. In which direction will a marble roll if placed on the surface at $x = 1$, $y = \frac{1}{2}$?

2. (i) Let $\phi(\mathbf{x})$ be a scalar field and $\mathbf{v}(\mathbf{x})$ a vector field. Show, using suffix notation, that

$$\nabla \cdot (\phi \mathbf{v}) = \nabla\phi \cdot \mathbf{v} + \phi \nabla \cdot \mathbf{v}, \qquad \nabla \times (\phi \mathbf{v}) = \nabla\phi \times \mathbf{v} + \phi \nabla \times \mathbf{v}.$$

(ii) Evaluate the divergence and curl of each of the following:

$$(\mathbf{a} \cdot \mathbf{x})\mathbf{b}, \quad \mathbf{a} \times \mathbf{x}, \quad r\mathbf{x}, \quad \frac{\mathbf{x} - \mathbf{a}}{|\mathbf{x} - \mathbf{a}|^3},$$

where $r = |\mathbf{x}|$ and $\mathbf{a}$, $\mathbf{b}$ are constant vectors.

(iii) The vector fields $\mathbf{F}$ and $\mathbf{G}$ are everywhere parallel, non-zero and solenoidal. Show that $\mathbf{F} \cdot \nabla(F/G) = 0$, where $F = |\mathbf{F}|$ and $G = |\mathbf{G}|$.

(iv) The vector field $\mathbf{B}(\mathbf{x})$ is everywhere parallel to the normals of a family of surfaces $f(\mathbf{x}) =$ constant. Show that $\mathbf{B} \cdot (\nabla \times \mathbf{B}) = 0$.

3. Verify directly that the vector field

$$\mathbf{u}(\mathbf{x}) = \big(e^x(x \cos y + \cos y - y \sin y), e^x(-x \sin y - \sin y - y \cos y), 0\big)$$

is irrotational and express it as the gradient of a scalar field ϕ. Check also that $\mathbf{u}$ is solenoidal and show that it can be written as the curl of a vector field $\mathbf{v} = (0, 0, \psi)$ for some function ψ.

4. Use suffix notation to show that for vector fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$,

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v}.$$

Show also that $(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla(\frac{1}{2}|\mathbf{u}|^2) - \mathbf{u} \times (\nabla \times \mathbf{u})$.

* **5.** Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a homogeneous function of degree n, i.e., $f(k\mathbf{x}) = k^n f(\mathbf{x})$ for all $k \in \mathbb{R}$. By differentiating with respect to k, or otherwise, show that $\mathbf{x} \cdot \nabla f = nf$.

* **6.** Suppose that $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a solenoidal vector field. Show that $\mathbf{F} = \nabla \times \mathbf{A}$ where

$$\mathbf{A}(\mathbf{x}) = \int_0^1 \mathbf{F}(t\mathbf{x}) \times (t\mathbf{x}) \, \mathrm{d}t.$$

This is an example of a *homotopy formula*, in this case for finding a vector potential for a given solenoidal field. What goes wrong if the domain of definition of $\mathbf{F}$ in $\mathbb{R}^3$ contains a hole of some kind?

Orthogonal Curvilinear Coordinates

7. If $\mathbf{a}$ is a constant vector and $r = |\mathbf{x}|$, verify that

$$\nabla r^n = nr^{n-2}\mathbf{x}, \qquad \nabla(\mathbf{a}\,.\,\mathbf{x}) = \mathbf{a}$$

using (i) Cartesian coordinates and suffix notation, (ii) Taylor's theorem, (iii) cylindrical polar coordinates, (iv) spherical polar coordinates. [*Hint: For parts (iii) and (iv) you will need to be careful about the components of* $\mathbf{a}$ *with respect to each of the relevant bases, which are* not *constant.*]

8. The vector field $\mathbf{A}(\mathbf{x})$ is, in Cartesian, cylindrical and spherical polar coordinates respectively,

$$\mathbf{A}(\mathbf{x}) = -\tfrac{1}{2}y\mathbf{e}_x + \tfrac{1}{2}x\mathbf{e}_y = \tfrac{1}{2}\rho\mathbf{e}_\phi = \tfrac{1}{2}r\sin\theta\,\mathbf{e}_\phi$$

(where $\mathbf{e}_\phi$ has two different meanings). Compute $\nabla\times\mathbf{A}$ in each coordinate system and check that your answers agree.

9. (i) Using the Chain Rule to express partial derivatives with respect to (x, y, z) in terms of partial derivatives with respect to cylindrical polar coordinates (ρ, ϕ, z), together with expressions for the basis vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$, show that for a function $f(\rho, \phi, z)$,

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z.$$

(ii) We deduce from this result that

$$\nabla = \mathbf{e}_\rho\frac{\partial}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z\frac{\partial}{\partial z}$$

and we also know that $\partial\mathbf{e}_\rho/\partial\phi = \mathbf{e}_\phi$, $\partial\mathbf{e}_\phi/\partial\phi = -\mathbf{e}_\rho$, while all other derivatives of the basis vectors are zero. Derive expressions for $\nabla\,.\,\mathbf{A}$ and $\nabla\times\mathbf{A}$ where $\mathbf{A}(\mathbf{x})$ is an arbitrary vector field given in cylindrical polars by $\mathbf{A} = A_\rho\mathbf{e}_\rho + A_\phi\mathbf{e}_\phi + A_z\mathbf{e}_z$.

* Also derive an expression for the Laplacian $\nabla^2 f$ of a scalar function $f(\rho, \phi, z)$.

Differential Geometry of Curves

10. Sketch the *astroid* curve in the plane given parametrically by $\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t), 0 \leqslant t \leqslant 2\pi$. Calculate $\dot{\mathbf{x}}(t)$ at each point and hence find the curve's total length.

11. A circular helix is given by

$$\mathbf{x}(t) = (a\cos t, a\sin t, ct), \quad t \in \mathbb{R}$$

where a, c are constants. Calculate the tangent $\mathbf{t}$, curvature κ, principal normal $\mathbf{n}$, binormal $\mathbf{b}$ and torsion τ. Give a sketch of the curve indicating the directions of the vectors $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$.

12. Show that a planar curve $\mathbf{x}(t) = \big(x(t), y(t), 0\big)$ has curvature

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

Use this result to find the minimum and maximum values of the curvature on the ellipse $x^2/a^2 + y^2/b^2 = 1$, and comment.

* **13.** The tangent vector at each point on a curve is parallel to a non-vanishing vector field $\mathbf{H}(\mathbf{x})$. Show that the curvature of the curve is given by $\kappa = |\mathbf{H}\times(\mathbf{H}\,.\,\nabla)\mathbf{H}|/|\mathbf{H}|^3$.