1. (a) The curve defined parametrically by $\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t)$ with $0 \leqslant t \leqslant 2\pi$ is called an *astroid*. Sketch it, and find its length.

   (b) The curve defined by $y^2 = x^3$ is called *Neile's parabola*. Sketch the segment of Neile's parabola with $0 \leqslant x \leqslant 4$, and find its length.

2. (a) A path in $\mathbb{R}^2$ is defined in polar coordinates by $r = f(\theta)$ for $\alpha \leqslant \theta \leqslant \beta$. Show that the length L of the path is given by

$$L = \int_\alpha^\beta \sqrt{\big(f(\theta)\big)^2 + \big(f'(\theta)\big)^2}\, \mathrm{d}\theta.$$

   (b) The curves $r = a\theta$, $r = ae^{b\theta}$ and $r = a(1 + \cos\theta)$ are called an *Archimedean spiral*, a *logarithmic spiral* and a *cardioid*, respectively. For $a, b > 0$ and $0 \leqslant \theta \leqslant 2\pi$, sketch the curves and find their lengths.

3. A circular helix is given by $\mathbf{x}(t) = (a\cos t, a\sin t, ct)$, where $a, c > 0$. Calculate the tangent $\mathbf{t}$, principal normal $\mathbf{n}$, curvature κ, binormal $\mathbf{b}$, and torsion τ. Sketch the helix, showing $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ at some point.

4. (a) Show that a planar curve given by $\mathbf{x}(t) = (x(t), y(t))$ has curvature

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

   (b) Find the maximum and minimum curvature of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, for $a > b > 0$.

5. Let f be a scalar function on $\mathbb{R}^2$. By expressing the Cartesian basis vectors $\mathbf{i}, \mathbf{j}$ in terms of the polar basis vectors $\mathbf{e}_\rho, \mathbf{e}_\phi$, and using the chain rule to express the partial derivatives with respect to Cartesian coordinates x, y in terms of those with respect to polar coordinates ρ, ϕ, show that

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi.$$

6. In three dimensions, use suffix notation and the summation convention to show that

$$\text{(i)}\ \ \boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a} \quad \text{and} \quad \text{(ii)}\ \ \boldsymbol{\nabla}(r^n) = nr^{n-2}\mathbf{x},$$

   where $\mathbf{a}$ is a constant vector and $r = |\mathbf{x}|$.

7. Find the equation of the plane tangent to the surface $z = 3x^2 y \sin(\frac{\pi}{2}x)$ at the point $x = y = 1$.

   Taking East to be in the direction $(1, 0, 0)$ and North to be $(0, 1, 0)$, in which compass direction will a marble roll if placed on the surface at $x = 1$, $y = \frac{1}{2}$?

8. Evaluate explicitly each of the line integrals

$$\text{(a)}\ \int (x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z), \qquad \text{(b)}\ \int (y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}z), \qquad \text{(c)}\ \int (y\,\mathrm{d}x - x\,\mathrm{d}y + e^{x+y}\,\mathrm{d}z)$$

   along (i) the straight line path joining $(0, 0, 0)$ to $(1, 1, 1)$, and (ii) the parabolic path given by $\mathbf{x}(t) = (t, t, t^2)$ with $0 \leqslant t \leqslant 1$.

   For the integrals that give the same answers, why does this happen?

9. Let $\mathbf{F}(\mathbf{x}) = (3x^2y^2z, 2x^3yz, x^3y^2)$ and $\mathbf{G}(\mathbf{x}) = (3x^2yz^2, 2x^3yz, x^3z^2)$ be vector fields.

   Show that $\mathbf{F}$ is conservative and $\mathbf{G}$ is not, and find the most general scalar potential for $\mathbf{F}$.

   By exploiting similarities between $\mathbf{F}$ and $\mathbf{G}$, evaluate the line integral $\int \mathbf{G} \cdot \mathrm{d}\mathbf{x}$ from $(0,0,0)$ to $(1,1,1)$ along the path $\mathbf{x}(t) = (t, \sin\frac{\pi}{2}t, \sin\frac{\pi}{2}t)$ with $0 \leqslant t \leqslant 1$.

10. (i) A curve C is given parametrically in Cartesian coordinates by

$$\mathbf{x}(t) = \big(\cos(\sin nt)\cos t,\ \cos(\sin nt)\sin t,\ \sin(\sin nt)\big), \quad 0 \leqslant t \leqslant 2\pi,$$

   where n is some fixed integer. Using spherical polar coordinates, sketch C.

   In case you haven't met them in lectures yet, spherical polars are defined by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, *with* $\theta \in [0,\pi]$ *and* $\phi \in [0, 2\pi)$.

   (ii) Let $\mathbf{F}(\mathbf{x}) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$. By evaluating the line integral explicitly, show that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 2\pi$, where C is traversed in the direction of increasing t.

   (iii) Find a scalar function f such that $\boldsymbol{\nabla} f = \mathbf{F}$. Comment on this, given (ii).

11. Use the substitution $x = \rho\cos\phi$, $y = \frac{1}{2}\rho\sin\phi$ to show that

$$\int_A \frac{x^2}{x^2+4y^2}\,\mathrm{d}A = \frac{3\pi}{4},$$

   where A is the region between the two ellipses $x^2 + 4y^2 = 1$ and $x^2 + 4y^2 = 4$.

12. The closed curve C in the (x,y) plane consists of the arc of the parabola $y^2 = 4ax$ $(a > 0)$ between the points $(a, \pm 2a)$ and the straight line joining $(a, \mp 2a)$. The region enclosed by C is A. By calculating both integrals explicitly, show that

$$\int_C (x^2y\,\mathrm{d}x + xy^2\,\mathrm{d}y) = \int_A (y^2 - x^2)\,\mathrm{d}A = \frac{104}{105}a^4,$$

   where C is traversed anticlockwise.

13. The region A is bounded by the straight line segments $\{x = 0, 0 \leqslant y \leqslant 1\}$, $\{y = 0, 0 \leqslant x \leqslant 1\}$, $\{y = 1, 0 \leqslant x \leqslant \frac{3}{4}\}$, and by an arc of the parabola $y^2 = 4(1-x)$. Consider a mapping into the (x,y) plane from the (u,v) plane defined by the transformation $x = u^2 - v^2$, $y = 2uv$. Sketch A and find the two regions in the (u,v) plane which are mapped into it.

   Hence calculate

$$\int_A \frac{\mathrm{d}A}{(x^2+y^2)^{1/2}}.$$

The remaining questions are optional.

14. A curve in the plane is given in polar coordinates as $r = f(\theta)$. Find an expression for its curvature as a function of θ.

    Find the curvature of the curve given by $r = \sin\theta$, and sketch it for $0 \leqslant \theta \leqslant \pi$.

15. The *field lines* of a non-vanishing vector field $\mathbf{F}$ are the curves parallel to $\mathbf{F}(\mathbf{x})$ at each point $\mathbf{x}$. Show that the curvature of the field lines of $\mathbf{F}$ is given by $|\mathbf{F}|^{-3}|\mathbf{F} \times (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{F}|$.

16. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a homogeneous function of degree n, i.e., such that $f(k\mathbf{x}) = k^n f(\mathbf{x})$ for all k. By differentiating with respect to k, show that $\mathbf{x} \cdot \boldsymbol{\nabla} f = nf$.

17. Without changing the order of integration, show that

$$\int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3} \, \mathrm{d}y \right) \mathrm{d}x = \frac{1}{2}, \quad \text{and} \quad \int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3} \, \mathrm{d}x \right) \mathrm{d}y = -\frac{1}{2}.$$

Comment on these results.

18. The curve $x^3 + y^3 - 3axy = 0$ with $a > 0$ is called the *Cartesian Leaf.* Sketch the Cartesian Leaf, and find the area bounded by it in the first quadrant.

Selected solutions for you to check your answers.

1. (b) The segment of Neile's parabola has length $\frac{16}{27}\big(10^{3/2} - 1\big)$.

2. (b) The lengths are as follows.

   - Archimedean spiral: $\frac{a}{2}(\sinh^{-1} 2\pi + 2\pi\sqrt{4\pi^2 + 1})$
   - logarithmic spiral: $\frac{a}{b}\sqrt{1 + b^2}(e^{2b\pi} - 1)$
   - cardioid: $8a$.

8. (a) both integrals are $\frac{3}{2}$; (b) both integrals are 2 ; (c) the integrals are $\frac{1}{2}(e^2 \mp 1)$

13. The area of A has the form $\lambda(1 + \log 2)$ for λ that you should work out.

1. Let T be the tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Find the volume V of T, and also the centre of volume, given by
$$\frac{1}{V}\int_T \mathbf{x}\,\mathrm{d}V\,.$$

2. A solid cone is bounded by the surface $\theta = \alpha$ (in spherical polar coordinates) and the surface $z = a$, and its density is $\rho_0 \cos\theta$. Show that its mass is $\frac{2}{3}\pi\rho_0 a^3(\sec\alpha - 1)$.

   You can use either spherical or cylindrical polars for this calculation. (You could do both!) It is also possible to use a mixture of spherical and cylindrical polar variables to make the limits of the integrals particularly nice.

3. Let a, b, c be positive. By using the new variables $\alpha = x/y$, $\beta = xy$, $\gamma = yz$, show that
$$\int_{x=0}^{\infty}\int_{y=0}^{1}\int_{z=0}^{x} x\,e^{-ax/y-bxy-cyz}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \;=\; \frac{1}{2a(a+b)(a+b+c)}.$$

4. (a) Let $\psi(\mathbf{x})$ be a scalar field and $\mathbf{v}(\mathbf{x})$ a vector field. Using suffix notation, show that
$$\boldsymbol{\nabla}\cdot(\psi\mathbf{v}) = (\boldsymbol{\nabla}\psi)\cdot\mathbf{v} + \psi\boldsymbol{\nabla}\cdot\mathbf{v} \quad\text{and}\quad \boldsymbol{\nabla}\times(\psi\mathbf{v}) = (\boldsymbol{\nabla}\psi)\times\mathbf{v} + \psi\boldsymbol{\nabla}\times\mathbf{v}.$$

   (b) Find the divergence and curl of the following vector fields on $\mathbb{R}^3$:
$$r\mathbf{x}, \qquad \mathbf{a}(\mathbf{b}\cdot\mathbf{x}), \qquad \mathbf{a}\times\mathbf{x}, \qquad \mathbf{x}/r^3,$$
   where $r = |\mathbf{x}|$, and $\mathbf{a}, \mathbf{b}$ are fixed vectors.

   Use part (a) where you can. Thinking about the definitions of divergence and curl, could you have guessed any of the answers in advance?

5. Let $\mathbf{u}$ and $\mathbf{v}$ be vector fields. Using suffix notation, show that

   (i) $\boldsymbol{\nabla}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\boldsymbol{\nabla}\times\mathbf{u}) - \mathbf{u}\cdot(\boldsymbol{\nabla}\times\mathbf{v})$

   (ii) $\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}(\boldsymbol{\nabla}\cdot\mathbf{v}) + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{u} - \mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{u}) - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v}$

   (iii) $\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{v}) + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v} + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{u}) + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{u}$

   Deduce from (iii) that $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}(\frac{1}{2}u^2) - \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})$, where $u = |\mathbf{u}|$.

   Feel free to omit any of these that were proved in lectures.

6. Verify that the vector field
$$\mathbf{v}(\mathbf{x}) = \big(e^x(x\cos y + \cos y - y\sin y), e^x(-x\sin y - \sin y - y\cos y), 0\big)$$
   is irrotational and express it as the gradient of a scalar field φ.

   Verify also that $\mathbf{v}$ is also solenoidal and express it as the curl of a vector field $(0,0,\psi)$.

7. (a) A vector field $\mathbf{F}$ is parallel to the normals of a family of surfaces $f(\mathbf{x}) = \text{constant}$. Show that $\mathbf{F}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) = 0$.

   (b) The vector fields $\mathbf{F}$ and $\mathbf{G}$ are everywhere parallel and are both solenoidal. Show that $\mathbf{F}\cdot\boldsymbol{\nabla}(F/G) = 0$, where $F = |\mathbf{F}|$ and $G = |\mathbf{G}| \neq 0$.

8. (a) *Revisiting sheet 1, question 6.*
Show that
$$\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a} \quad \text{and} \quad \boldsymbol{\nabla}(r^n) = nr^{n-2}\mathbf{x},$$
where $\mathbf{a}$ is a constant vector and $r = |\mathbf{x}|$, using (i) cylindrical polar coordinates, and (ii) spherical polar coordinates. *Hint: choose axes to make* $\mathbf{a}$ *nice.*

(b) Calculate, in three ways, the curl of the vector field
$$\mathbf{F}(\mathbf{x}) \;=\; -y\mathbf{e}_x + x\mathbf{e}_y \;=\; \rho\mathbf{e}_\phi \;=\; r\sin\theta\,\mathbf{e}_\phi,$$
by applying the standard formulae in Cartesian, cylindrical, and spherical coordinates.
By considering the relationship between the basis vectors, check that your answers agree.

9. Show that the unit basis vectors of cylindrical polar coordinates satisfy
$$\frac{\partial \mathbf{e}_\rho}{\partial\phi} = \mathbf{e}_\phi \quad \text{and} \quad \frac{\partial \mathbf{e}_\phi}{\partial\phi} = -\mathbf{e}_\rho$$
and that all other derivatives of the three basis vectors are zero.
Given that the operator $\boldsymbol{\nabla}$ in cylindrical polars is
$$\boldsymbol{\nabla} = \mathbf{e}_\rho\frac{\partial}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z\frac{\partial}{\partial z},$$
derive expressions for $\boldsymbol{\nabla}\cdot\mathbf{F}$ and $\boldsymbol{\nabla}\times\mathbf{F}$, where $\mathbf{F} = F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z$.
Also derive an expression for $\nabla^2 f$, for a scalar function f.
Don't just quote div/curl formulae from notes – the intention is for you to derive them here.

10. Let $I = \int_S r^n\,\mathbf{x}\cdot \mathrm{d}\mathbf{S}$, for $n > 0$, where $r = |\mathbf{x}|$ and S is the sphere of radius R centred at the origin in $\mathbb{R}^3$. Evaluate I directly, and by means of the divergence theorem.

11. Let $\mathbf{F}(\mathbf{x}) = (x^3 + 3y + z^2,\, y^3,\, x^2 + y^2 + 3z^2)$, and let S be the *open* surface
$$x^2 + y^2 = 1 - z, \quad 0 \leqslant z \leqslant 1.$$
Use the divergence theorem (and cylindrical polar coordinates) to show that $\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = 2\pi$.
Verify this by direct calculation.

12. By applying the divergence theorem to the vector field $\mathbf{c}\times\mathbf{F}$, where $\mathbf{c}$ is an arbitrary constant vector and $\mathbf{F}(\mathbf{x})$ is a vector field, show that
$$\int_V \boldsymbol{\nabla}\times\mathbf{F}\;\mathrm{d}V = \int_A \mathrm{d}\mathbf{A}\times\mathbf{F}\,,$$
where the surface A encloses the volume V.
Verify this result when $\mathbf{F}(\mathbf{x}) = (z, 0, 0)$ and V is the cuboid $0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b,\ 0 \leqslant z \leqslant c$.

13. Let V be a volume with boundary ∂V, and let f be a scalar field and $\mathbf{F}$ a vector field on V. By applying the divergence theorem to suitable vector fields, prove the following results.

(i) If f is constant on ∂V, then $\int_V \boldsymbol{\nabla} f\;\mathrm{d}V = \mathbf{0}$.
(ii) If $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$ in V and $\mathbf{F}\cdot\mathbf{n} = 0$ on ∂V, then $\int_V \mathbf{F}\;\mathrm{d}V = \mathbf{0}$.

Explain why these results make sense, given the conditions on f and $\mathbf{F}$.

The remaining questions are optional.

14. A *tricylinder* is the body formed by intersecting the three solid cylinders given by the equations $x^2 + y^2 \leqslant a^2$, $y^2 + z^2 \leqslant a^2$ and $z^2 + x^2 \leqslant a^2$, with $a > 0$. Using cylindrical polar coordinates, show that the volume of a tricylinder is $8(2 - \sqrt{2})a^3$.

15. Using the formula for $\boldsymbol{\nabla}$ in the appropriate coordinates, prove the following two results.

    (i) Let L be the line through the origin in the direction of the fixed non-zero vector $\mathbf{a}$. Then the field $f(\mathbf{x})$ has cylindrical symmetry about L if and only if $\mathbf{a} \cdot (\mathbf{x} \times \boldsymbol{\nabla} f) = 0$.

    (ii) The field $f(\mathbf{x})$ has spherical symmetry about the origin if and only if $\mathbf{x} \times \boldsymbol{\nabla} f = \mathbf{0}$.

    Explain geometrically why these results make sense.

16. (a) By considering
$$\mathbf{G}(\mathbf{x}) = \int_0^1 \mathbf{F}(t\mathbf{x}) \times t\mathbf{x} \, \mathrm{d}t,$$
show that if $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$ then $\boldsymbol{\nabla} \times \mathbf{G} = \mathbf{F}$.

    (b) By considering
$$g(\mathbf{x}) = \int_0^1 \mathbf{F}(t\mathbf{x}) \cdot \mathbf{x} \, \mathrm{d}t,$$
show that if $\mathbf{x} \times (\boldsymbol{\nabla} \times \mathbf{F}) = \mathbf{0}$ then $\boldsymbol{\nabla} g = \mathbf{F}$.

    Show that $\mathbf{F}(\mathbf{x}) = \mathbf{c} \times \mathbf{x}/r^2$, where $\mathbf{c}$ is a non-zero constant vector, satisfies $\boldsymbol{\nabla} \times \mathbf{F} \neq \mathbf{0}$ and $\mathbf{x} \times (\boldsymbol{\nabla} \times \mathbf{F}) = \mathbf{0}$. Why does this not contradict the usual necessary and sufficient condition for $\mathbf{F}$ to be the gradient of a scalar?

Selected solutions for you to check your answers.

4. (b) The divergences are: $4r$, $\mathbf{a} \cdot \mathbf{b}$, 0, 0. The curls are: $\mathbf{0}$, $\mathbf{b} \times \mathbf{a}$, $2\mathbf{a}$, $\mathbf{0}$.

6. Interesting fact: it turns out that $\phi + i\psi = ze^z$, where $z = x + iy$.

12. The integrals equal $(0, abc, 0)$.

1. The vector field $\mathbf{F}$ is given in cylindrical polar coordinates (ρ, ϕ, z) by $\mathbf{F} = \frac{1}{\rho}\mathbf{e}_\phi$, for $\rho \neq 0$. Show that $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$ for $\rho \neq 0$. Calculate $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$ with C the circle given by $\rho = R$, $0 \leqslant \phi \leqslant 2\pi$, $z = 0$. Why does Stokes' theorem not apply?

2. Verify Stokes' theorem for the vector field $\mathbf{F}(\mathbf{x}) = (y, -x, xyz)$ and the open surface defined in cylindrical polar coordinates by $\rho + z = a$ and $z \geqslant 0$, where $a > 0$.

3. Verify Stokes' theorem for the vector field $\mathbf{F}(\mathbf{x}) = (-y^3, x^3, z^3)$ and the open surface defined by $z = x^2 + y^2$ and $\frac{1}{4} \leqslant z \leqslant 1$.

4. By applying Stokes' theorem to the vector field $\mathbf{c} \times \mathbf{F}$, where $\mathbf{c}$ is an arbitrary constant vector and $\mathbf{F}$ is a vector field, show that

$$\oint_C \mathrm{d}\mathbf{x} \times \mathbf{F} = \int_A (\mathrm{d}\mathbf{A} \times \boldsymbol{\nabla}) \times \mathbf{F}\,,$$

   where the curve C bounds the open surface A.

   Verify this result when $\mathbf{F}(\mathbf{x}) = \mathbf{x}$ and A is the unit square in the (x, y) plane with opposite vertices at $(0, 0, 0)$ and $(1, 1, 0)$.

5. Let $\mathbf{F}, \mathbf{G}$ be vector fields satisfying $\boldsymbol{\nabla} \cdot \mathbf{F} = \boldsymbol{\nabla} \cdot \mathbf{G} = 0$ in the volume V. Show that

$$\int_V \mathbf{F} \cdot \nabla^2 \mathbf{G} - \mathbf{G} \cdot \nabla^2 \mathbf{F} \, \mathrm{d}V = \int_{\partial V} \big(\mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) - \mathbf{G} \times (\boldsymbol{\nabla} \times \mathbf{F})\big) \cdot \mathrm{d}\mathbf{S}.$$

6. (a) The scalar field $\varphi(r)$ depends only on the radial distance $r = |\mathbf{x}|$ in $\mathbb{R}^3$. Use Cartesian coordinates and the chain rule to show that

$$\boldsymbol{\nabla}\varphi = \varphi'(r)\frac{\mathbf{x}}{r} \quad \text{and} \quad \nabla^2\varphi = \varphi''(r) + \frac{2}{r}\varphi'(r).$$

   What are the corresponding results when working in $\mathbb{R}^2$ rather than $\mathbb{R}^3$?

   (b) Show that the radially symmetric solutions of Laplace's equation in $\mathbb{R}^2$ have the form $\varphi = \alpha + \beta \log r$, where α and β are constants.

   (c) Find the solution of $\nabla^2\varphi = 1$ in the region $r \leqslant 1$ in $\mathbb{R}^3$ which is not singular at the origin and satisfies $\varphi(1) = 1$.

7. A spherically symmetric charge density is given by

$$\rho(r) = \begin{cases} 0 & \text{for } 0 \leqslant r < a \\ r & \text{for } a \leqslant r \leqslant b \\ 0 & \text{for } b < r < \infty \end{cases}$$

   Find the electric field everywhere in two ways, namely:

   (i) direct solution of Poisson's equation, using formulae from question 6,

   (ii) Gauss's flux method.

   You should assume that the potential is a function only of r, is not singular at the origin and that the potential and its first derivative are continuous at $r = a$ and $r = b$.

   (Note that you are asked to find the electric field, not the potential.)

8. (i) Find all solutions of Laplace's equation, $\nabla^2 f = 0$, in two dimensions that can be written in the separable form $f(r,\theta) = R(r)\Phi(\theta)$, where r and θ are plane polar coordinates.

   (ii) Consider the following boundary value problem in $\mathbb{R}^2$.

$$\nabla^2 f = 0, \quad f(a,\theta) = \sin\theta.$$

   Find the solution for $r \leqslant a$ which is not singular at the origin.
   Find the solution for $r \geqslant a$ that satisfies $f(r,\theta) \to 0$ as $r \to \infty$.
   Find the solution for $a \leqslant r \leqslant b$ that satisfies $\frac{\partial f}{\partial n}(b,\theta) = 0$. *Recall that* $\frac{\partial f}{\partial n} = \mathbf{n} \cdot \boldsymbol{\nabla} f$.

9. The surface S encloses a volume in which the scalar field φ satisfies the Klein-Gordon equation $\nabla^2\varphi = m^2\varphi$, where m is a real non-zero constant. Prove that φ is uniquely determined if either φ or $\partial\varphi/\partial n$ is given on S.

10. Show that the solution to Laplace's equation $\nabla^2\varphi = 0$ in a volume V with boundary condition

$$g\,\frac{\partial\varphi}{\partial n} + \varphi = f \quad \text{on} \quad \partial V$$

   is unique if $g(\mathbf{x}) \geqslant 0$ on ∂V.

   Find a non-zero (and so non-unique) solution of Laplace's equation defined on $r \leqslant 1$ which satisfies the boundary condition above with $f = 0$ and $g = -1$ on $r = 1$.

   Don't assume that the solution is spherically symmetric. (Why not?)

11. The scalar fields $u(\mathbf{x})$ and $v(\mathbf{x})$ satisfy $\nabla^2 u = 0$ on V and $v = 0$ on ∂V . Show that

$$\int_V \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \; \mathrm{d}V = 0.$$

   Let w be a scalar field which satisfies $w = u$ on ∂V. Show that

$$\int_V |\boldsymbol{\nabla} w|^2 \; \mathrm{d}V \geqslant \int_V |\boldsymbol{\nabla} u|^2 \; \mathrm{d}V,$$

   i.e. the solution of the Laplace problem minimises $\int_V |\boldsymbol{\nabla} w|^2 \, \mathrm{d}V$.

12. The scalar field φ is harmonic (i.e., solves Laplace's equation) in a volume V bounded by a closed surface S. Given that V does not contain the origin, show that

$$\int_S \left(\varphi\boldsymbol{\nabla}\left(\frac{1}{r}\right) - \left(\frac{1}{r}\right)\boldsymbol{\nabla}\varphi\right) \cdot \mathrm{d}\mathbf{S} = 0.$$

   Now let V be the volume given by $\varepsilon \leqslant r \leqslant a$ and let S_a be the surface $r = a$. Given that $\varphi(\mathbf{x})$ is harmonic for $r \leqslant a$, use this result, in the limit $\varepsilon \to 0$, to show that

$$\varphi(0) = \frac{1}{4\pi a^2}\int_{S_a} \varphi(\mathbf{x}) \; \mathrm{d}S.$$

The remaining questions are optional.

13. Find the electric field in question 7 by a third method, namely the integral solution of Poisson's equation: if $\nabla^2\varphi = \rho$ then
$$\varphi(\mathbf{y}) = -\frac{1}{4\pi}\int_V \frac{\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}\ \mathrm{d}V.$$

14. Let S be a surface with unit normal $\mathbf{n}$, and $\mathbf{v}(\mathbf{x})$ a vector field such that $\mathbf{v}\cdot\mathbf{n} = 0$ on S. Let $\mathbf{m}$ be a unit vector field such that $\mathbf{m} = \mathbf{n}$ on S. By applying Stokes' theorem to $\mathbf{m}\times\mathbf{v}$, show that
$$\int_S (\delta_{ij} - n_i n_j)\,\frac{\partial v_i}{\partial x_j}\ \mathrm{d}S = \oint_C \mathbf{u}\cdot\mathbf{v}\ \mathrm{d}s,$$
where s denotes arc-length along the boundary C of S, and $\mathbf{u}$ is such that $\mathbf{u}\mathrm{d}s = \mathrm{d}\mathbf{s}\times\mathbf{n}$.

    Verify this result by taking $\mathbf{v} = \mathbf{x}$ and S to be the disc $|\mathbf{x}| \leqslant R$ in the $z = 0$ plane.

15. Let S_1 be the 3-dimensional sphere of radius 1 centred at $(0,0,0)$, S_2 be the sphere of radius $\frac{1}{2}$ centred at $(\frac{1}{2},0,0)$ and S_3 be the sphere of radius $\frac{1}{4}$ centred at $(-\frac{1}{4},0,0)$.

    The eccentrically-shaped planet Zog is composed of rock of uniform density ρ occupying the region within S_1 and outside S_2 and S_3. The regions inside S_2 and S_3 are empty. Give an expression for Zog's gravitational potential at a general coordinate $\mathbf{x}$ that is outside S_1.

    Show that there is a point in the interior of S_3 where a particle would remain at rest. Does it do so stably?

16. Consider the partial differential equation $\dfrac{\partial u}{\partial t} = \nabla^2 u$, for $u = u(t,\mathbf{x})$, with initial condition $u(0,\mathbf{x}) = u_0(\mathbf{x})$ in V, and boundary condition $u(t,\mathbf{x}) = f(\mathbf{x})$ on ∂V for all $t \geqslant 0$.

    Show that $\dfrac{d}{dt}\displaystyle\int_V |\boldsymbol{\nabla} u|^2\ \mathrm{d}V \leqslant 0$. When does equality hold?

Selected solutions for you to check your answers.

3. The integrals both equal $\frac{45}{32}\pi$ (or both equal $-\frac{45}{32}\pi$, depending on your choice of orientation).

6. (c) The solution is $\phi(r) = \frac{1}{6}(r^2+5)$.

7. The field is given by $\mathbf{E}(r) = E(r)\mathbf{e}_r$, where $E(r) = \begin{cases} 0 & \text{for } 0 \leqslant r < a \\ (r^4-a^4)/4\varepsilon_0 r^2 & \text{for } a \leqslant r \leqslant b \\ (b^4-a^4)/4\varepsilon_0 r^2 & \text{for } b < r < \infty \end{cases}$

8. (b) The three solutions are, respectively: $\frac{r}{a}\sin\theta$, $\frac{a}{r}\sin\theta$, $\frac{a}{a^2+b^2}\left(r+\frac{b^2}{r}\right)\sin\theta$.

We are working in $\mathbb{R}^3$ throughout. Where needed, you may quote that the most general isotropic tensor of rank 4 has the form $\lambda\delta_{ij}\delta_{k\ell} + \mu\delta_{ik}\delta_{j\ell} + \nu\delta_{i\ell}\delta_{jk}$ for $\lambda, \mu, \nu \in \mathbb{R}$.

1. Any 3×3 matrix A can be decomposed in the form $A\mathbf{x} = \alpha\mathbf{x} + \mathbf{v} \times \mathbf{x} + B\mathbf{x}$, where α is a scalar, $\mathbf{v}$ is a vector, and B is a traceless symmetric matrix. Verify this claim by finding α, v_k and B_{ij} explicitly in terms of A_{ij}.

   Find α, $\mathbf{v}$ and B for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$.

2. Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^3$, and let $P_{ij} = \delta_{ij} - n_i n_j$.

   (a) Find the eigenvalues and eigenvectors of P_{ij}.

   (b) For $A_{ij} = \varepsilon_{ijk} n_k$, find $A_{ij}A_{jk}A_{k\ell}A_{\ell m}$ in terms of P_{im}. Explain this result geometrically.

3. Given vectors $\mathbf{u} = (1, 0, 1)$, $\mathbf{v} = (0, 1, -1)$ and $\mathbf{w} = (1, 1, 0)$, find all components of the second-rank and third-rank tensors defined by

   (i) $S_{ij} = u_i v_j + v_i w_j$

   (ii) $T_{ijk} = u_i v_j w_k - v_i u_j w_k + v_i w_j u_k - w_i v_j u_k + w_i u_j v_k - u_i w_j v_k$.

4. If $\mathbf{u}(\mathbf{x})$ is a vector field, show that $\partial u_i / \partial x_j$ transforms as a second-rank tensor.

   If $\sigma(\mathbf{x})$ is a second-rank tensor field, show that $\partial \sigma_{ij} / \partial x_j$ transforms as a vector.

5. Let $\mathbf{u}(\mathbf{x})$ be a vector field on $\mathbb{R}^3$. Show that

   $$\frac{\partial u_i}{\partial x_j} = \tfrac{1}{3}(\boldsymbol{\nabla} \cdot \mathbf{u})\delta_{ij} + S_{ij} - \tfrac{1}{2}\varepsilon_{ijk}(\boldsymbol{\nabla} \times \mathbf{u})_k\ ,$$

   for some traceless symmetric tensor S_{ij}.

   Find S_{ij} in the case $\mathbf{u}(\mathbf{x}) = (x_1 x_2^2,\ x_2 x_3^2,\ x_3 x_1^2)$. Verify that $(0, 0, 1)$ is one of the principal axes of S_{ij} at the point $\mathbf{x} = (2, 3, 0)$, and find the others.

6. The current J_i due to an electric field E_i is given by $J_i = \sigma_{ij} E_j$, where σ_{ij} is the conductivity tensor. In a given Cartesian coordinate system

   $$(\sigma_{ij}) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

   Show that for one direction of the applied electric field, no current flows. For which direction(s) of E_i is the current largest?

7. An object has symmetry such that it is unchanged by rotations of π about the three usual Cartesian axes. Show that any second-rank tensor calculated for the object will be diagonal in the standard basis, although the diagonal elements need not be equal.

   Find the inertia tensor of a cuboid of uniform density ρ with sides of length $2a$, $2b$ and $2c$ about the centre of the cuboid.

8. For any second-rank tensor T_{ij}, prove using the transformation law that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ij}T_{ji}, \quad \text{and} \quad \gamma = T_{ij}T_{jk}T_{ki}$$

are the same in all bases.

If T_{ij} is a symmetric tensor, express these invariants in terms of the eigenvalues. Deduce that the determinant of T_{ij} is $\frac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma)$.

9. Let S be the surface of the unit sphere.

(a) Calculate the following integrals using properties of isotropic tensors:

$$\text{(i)} \quad \int_S x_i \, \mathrm{d}S, \qquad \text{(ii)} \quad \int_S x_i x_j \, \mathrm{d}S.$$

Verify your answers using the tensor divergence theorem (if you have met it in lectures).

(b) For the second-rank tensor $T_{ij} = \delta_{ij} + \varepsilon_{ijk}x_k$, calculate the following integrals:

$$\text{(i)} \quad \int_S T_{ij} \, \mathrm{d}S, \qquad \text{(ii)} \quad \int_S T_{ij}T_{jk} \, \mathrm{d}S.$$

If your lectures only considered isotropic integrals over volumes rather than surfaces, then you should assume that the corresponding results for surfaces hold.

10. Evaluate the following integrals over the whole of $\mathbb{R}^3$.

$$\text{(i)} \int r^{-3}e^{-r^2} x_i x_j \, \mathrm{d}V, \quad \text{(ii)} \int r^{-4}e^{-r^2} x_i x_j x_k \, \mathrm{d}V, \quad \text{(iii)} \int r^{-5}e^{-r^2} x_i x_j x_k x_\ell \, \mathrm{d}V.$$

11. (a) Show that $\varepsilon_{ijk}\varepsilon_{ij\ell}$ is isotropic of rank 2, and deduce that it equals $2\delta_{k\ell}$.

(b) Using that $\varepsilon_{ijk}\varepsilon_{i\ell m}$ is isotropic of rank 4, show that it equals $\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}$.

(c) Prove that $\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}$.

12. The array d_{ijk} with 3^3 elements is such that $d_{ijk}s_{jk}$ is a vector for every symmetric second-rank tensor s_{jk}. Show that d_{ijk} need not be a tensor, but that $d_{ijk} + d_{ikj}$ must be.

13. In linear elasticity, the symmetric second-rank stress tensor σ_{ij} depends on the symmetric second-rank strain tensor e_{ij} according to $\sigma_{ij} = c_{ijk\ell}e_{k\ell}$, where $c_{ijk\ell}$ is a fourth-rank tensor. Show that, in an isotropic material,

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} \qquad (*)$$

for two scalars λ and μ.

Assume now that $\mu > 0$ and $\lambda > -\frac{2}{3}\mu$.

Use $(*)$ to find an expression for e_{ij} in terms of σ_{ij}, and explain why the principal axes of σ_{ij} and e_{ij} coincide.

The elastic energy density resulting from a deformation of the material is given by $E = \frac{1}{2}e_{ij}\sigma_{ij}$. Show that $E > 0$ for any non-zero strain e_{ij}.

The remaining questions are optional.

14. Let v_i be a non-zero vector. Show that any 3×3 symmetric matrix T_{ij} can be expressed in the form $T_{ij} = A\delta_{ij} + Bv_iv_j + (C_iv_j + C_jv_i) + D_{ij}$ for scalars A and B, a vector C_i satisfying $C_iv_i = 0$, and a symmetric matrix D_{ij} satisfying $D_{ii} = 0$ and $D_{ij}v_j = 0$.

    Explain why A, B, C_i and D_{ij} together provide a space of the correct dimension to parameterise an arbitrary symmetric 3×3 matrix T_{ij}.

15. (a) A tensor of rank 3 satisfies $T_{ijk} = T_{jik}$ and $T_{ijk} = -T_{ikj}$. Show that $T_{ijk} = 0$.

    (b) A tensor of rank 4 satisfies $T_{jik\ell} = -T_{ijk\ell} = T_{ij\ell k}$ and $T_{ijij} = 0$. Show that

    $$T_{ijk\ell} = \varepsilon_{ijp}\varepsilon_{k\ell q}S_{pq}, \quad \text{where} \quad S_{pq} = -T_{rqrp}.$$

16. (a) In question 9, you were told to assume that the results from lectures regarding isotropic integrals over volumes also hold for surfaces. Prove that this is valid for a spherical surface, using the tensor divergence theorem.

    (b) Using the tensor divergence theorem, verify the formula in question 12 on sheet 2:

    $$\int_V \boldsymbol{\nabla} \times \mathbf{F} \, \mathrm{d}V = \int_A \mathrm{d}\mathbf{S} \times \mathbf{F} \, .$$

    (c) Using a 'tensor Stokes' theorem', verify the formula in question 4 on sheet 3:

    $$\oint_C \mathrm{d}\mathbf{x} \times \mathbf{F} = \int_A (\mathrm{d}\mathbf{A} \times \boldsymbol{\nabla}) \times \mathbf{F} \, .$$

17. A second-rank tensor $T(\mathbf{y})$ is defined for $n > -1$ by

    $$T_{ij}(\mathbf{y}) = \int_S (y_i - x_i)(y_j - x_j)|\mathbf{y} - \mathbf{x}|^{2n-2} \, \mathrm{d}A \, ,$$

    where $\mathbf{y}$ is a fixed unit vector and S is the unit sphere.

    Given that $T_{ij} = \alpha\delta_{ij} + \beta y_iy_j$ for some scalar constants α and β, find the value of n for which T is isotropic.

Selected solutions for you to check your answers.

5. The other principal axes are $(2, 1, 0)$ and $(1, -2, 0)$.

7. The inertia tensor has diagonal entries $\frac{M}{3}(b^2 + c^2)$, $\frac{M}{3}(c^2 + a^2)$, $\frac{M}{3}(a^2 + b^2)$, where $M = 8abc\rho$.

9. The answers are: (a)(i) 0, (a)(ii) $\frac{4}{3}\pi\delta_{ij}$, (b)(i) $4\pi\delta_{ij}$, (b)(ii) $\frac{4}{3}\pi\delta_{ik}$.

10. The answers are: (i) $\frac{2}{3}\pi\delta_{ij}$, (ii) 0, (iii) $\frac{2\pi}{15}(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$.