Vector Calculus: Example Sheet 4

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1. The current $\mathbf{J}$ due to an electric field $\mathbf{E}$ is given by $J_i = \sigma_{ij} E_j$, where σ_{ij} is the conductivity tensor. In a given Cartesian coordinate system

$$\sigma_{ij} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Show that there is a direction along which no current flows, and find the direction(s) along which the current is largest, for an electric field of fixed magnitude.

2. Given the vectors $\mathbf{u} = (1, 0, 1)$, $\mathbf{v} = (0, 1, -1)$ and $\mathbf{w} = (1, 1, 0)$, find all components of the second-rank and third-rank tensors defined by

$$T_{ij} = u_i v_j + v_i w_j; \quad S_{ijk} = u_i v_j w_k - v_i u_j w_k + v_i w_j u_k - w_i v_j u_k + w_i u_j v_k - u_i w_j v_k.$$

3. Use the transformation law for a second-rank tensor T_{ij}, show that the quantities

$$\alpha = T_{ii}, \quad \beta = T_{ij} T_{ji}, \quad \gamma = T_{ij} T_{jk} T_{ki}$$

are scalars, i.e. remain the same in all Cartesian coordinate systems. If T_{ij} is diagonal in some coordinate system, express the quantities above in terms of its eigenvalues. Hence deduce that the eigenvalues are the roots of the cubic equation

$$\lambda^3 - \alpha\lambda^2 + \frac{1}{2}(\alpha^2 - \beta)\lambda - \frac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0.$$

4. If $u_i(\mathbf{x})$ is a vector field, show that $\partial u_i / \partial x_j$ transforms as a second rank tensor field. If $\sigma_{ij}(\mathbf{x})$ is a tensor field, show that $\partial \sigma_{ij} / \partial x_j$ transforms as a vector field.

5. The electric field $\mathbf{E}(\mathbf{x}, t)$ and magnetic field $\mathbf{B}(\mathbf{x}, t)$ satisfy Maxwell's equations with zero charge and current. Show that the Poynting vector $\mathbf{P} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$ satisfies the conservation law

$$\frac{\partial P_i}{\partial t} + \frac{\partial T_{ij}}{\partial x_j} = 0 \quad \text{where} \quad T_{ij} = \frac{1}{\mu_0}\left(\frac{1}{2}\delta_{ij}(\mathbf{E}^2 + c^2\mathbf{B}^2) - (E_i E_j + c^2 B_i B_j)\right)$$

where $c^2 = 1/\mu_0\epsilon_0$. If the component P_i is the momentum density in the x^i direction stored in the electric and magnetic fields, what is the interpretation of T_{ij}?

6. The velocity field $\mathbf{u}(\mathbf{x},t)$ of an inviscid compressible gas obeys

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \quad \text{and} \quad \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla P$$

where $\rho(\mathbf{x},t)$ is the density and $P(\mathbf{x},t)$ is the pressure. Show that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_j u_j\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho u_j u_j u_i + P u_i\right) = P\nabla\cdot\mathbf{u} \quad \text{and} \quad \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial t_{ij}}{\partial x_j} = 0$$

for a suitable symmetric tensor t_{ij}, to be determined.

7. By first decomposing into symmetric and anti-symmetric parts, show that an arbitrary second rank tensor T_{ij} can be written in the form

$$T_{ij} = \alpha\delta_{ij} + \epsilon_{ijk}\omega_k + B_{ij}$$

where α is a scalar, ω_k a vector and B_{ij} a symmetric second rank tensor satisfying $B_{ii} = 0$.

(i) A tensor of rank 3 satisfies $T_{ijk} = T_{jik}$ and $T_{ijk} = -T_{ikj}$. Show that $T_{ijk} = 0$.

(ii) A tensor of rank 4 satisfies $T_{jikl} = -T_{ijkl} = T_{ijlk}$ and $T_{ijij} = 0$. Show that

$$T_{ijkl} = \epsilon_{ijp}\epsilon_{klq}S_{pq} \quad \text{where} \quad S_{pq} = -T_{rqrp}.$$

8. A cuboid of uniform density and mass M has sides of length $2a, 2b$ and $2c$. Find the inertia tensor about its centre, with respect to a coordinate system of your choice.

A cube of sides of length $2a$ has uniform charge density, mass M, and is rotating with angular velocity $\boldsymbol{\omega}$ about an axis which passes through its center and through a pair of opposite vertices. What is its angular momentum?

9. Evaluate the following integrals, where $\gamma > 0$ and $r^2 = x_p x_p$:

$$\text{(i)} \int_{\mathbb{R}^3} r^{-3}e^{-\gamma r^2}x_i x_j\,\mathrm{d}V, \quad \text{(ii)} \int_{\mathbb{R}^3} r^{-5}e^{-\gamma r^2}x_i x_j x_k\,\mathrm{d}V.$$

10. A tensor has components T_{ij} with respect to a given Cartesian coordinate system $\{x_i\}$. If the tensor is invariant under arbitrary rotations about the x_3-axis, show that it must have the form

$$T_{ij} = \begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

11. In the theory of linear elasticity, the symmetric stress tensor σ_{ij} depends on the symmetric strain tensor e_{kl} through the equation $\sigma_{ij} = C_{ijkl}e_{kl}$. Explain why C_{ijkl} must be a fourth rank tensor, assuming that $C_{ijkl} = C_{ijlk}$. For an isotropic medium, use the most general possible form for C_{ijkl} (which you may quote) to show that

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij},$$

where λ and μ are scalars. Invert this equation to express e_{ij} in terms of σ_{ij}, assuming $\mu \neq 0$ and $3\lambda \neq -2\mu$. Explain why the principal axes of σ_{ij} and e_{ij} coincide.

The elastic energy density resulting from a deformation of the medium is $E = \frac{1}{2}e_{ij}\sigma_{ij}$. Show that E is strictly positive for any non-zero strain e_{ij} provided $\mu > 0$ and $\lambda > -\frac{2}{3}\mu$.

12*. The totally anti-symmetric tensor of rank n is defined by

$$\epsilon_{i_1 i_2 \cdots i_n} = \begin{cases} +1, & \text{if } (i_1, i_2, \ldots, i_n) \text{ is an even permutation of } (1, 2, \ldots, n), \\ -1, & \text{if } (i_1, i_2, \ldots, i_n) \text{ is an odd permutation of } (1, 2, \ldots, n), \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\epsilon_{i_1 i_2 \cdots i_n}\epsilon_{i_1 i_2 \cdots i_n} = n!$.

How might you get a computer to compute $\epsilon_{i_1 \cdots i_n}$ for a given permutation $(i_1, \ldots, i_n)$ of $(1, , \ldots, n)$? For instance, computing the determinant of the relevant permutation matrix is one option, but is quite computationally complex. Can you do better?

13*. Let $T_{ij\cdots k}$ be a tensor of rank m in $\mathbb{R}^n$. How many independent components does $T_{ij\cdots k}$ have if it is (a) totally antisymmetric; or (b) totally symmetric?