Vector Calculus: Example Sheet 3

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We will have covered the material necessary to attempt all these questions by the end of lecture 19.

1. Consider the line integral

$$I = \oint_C -x^2 y \, \mathrm{d}x + xy^2 \, \mathrm{d}y$$

for C a closed curve traversed anti-clockwise in the (x, y)-plane.

(i) Evaluate I when C is a circle of radius R centred at the origin. Use Green's theorem to relate the results for $R = b$ and $R = a$ to an area integral over an appropriate region, and calculate the area integral directly.

(ii) Now suppose C is the boundary of a square centred at the origin with sides of length ℓ. Show that I does not change if the square is rotated in the (x, y)-plane.

2. Verify Stokes' theorem for the hemispherical shell $S = \{x^2 + y^2 + z^2 = 1, z \geq 0\}$, and the vector field

$$\mathbf{F}(\mathbf{x}) = (y, -x, z).$$

3. By applying Stokes' theorem to the vector field $\mathbf{a} \times \mathbf{F}$ for $\mathbf{a}$ constant, or otherwise, show that for a vector field $\mathbf{F}(\mathbf{x})$

$$\oint_C \mathrm{d}\mathbf{x} \times \mathbf{F} = \int_S (\mathrm{d}\mathbf{S} \times \nabla) \times \mathbf{F}$$

where $C = \partial S$. Verify this result when C is the boundary of a unit square lying in the (x, y)-plane, with opposite vertices at $(0, 0, 0)$ and $(1, 1, 0)$, and $\mathbf{F}(\mathbf{x}) = \mathbf{x}$.

4. Let $S = \{\mathbf{x} : |\mathbf{x}| = 1\}$ be the surface of a unit sphere. For the vector field

$$\mathbf{F}(\mathbf{x}) = \frac{\mathbf{x}}{r^3}$$

where $r = |\mathbf{x}|$, compute the integral $\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$. Deduce that there *does not* exist a vector potential for $\mathbf{F}$, i.e. there can be no $\mathbf{A}$ for which $\mathbf{F} = \nabla \times \mathbf{A}$. Compute $\nabla \cdot \mathbf{F}$ and comment on your result.

5*. Consider the following vector field

$$\mathbf{A}(\mathbf{x}) = \frac{1}{(x^2+y^2)r}\,(yz, -xz, 0)$$

where $r = |\mathbf{x}|$. Compute $\nabla \times \mathbf{A}$. Does this contradict the result of Question 4? Apply Stokes' theorem to $\nabla \times \mathbf{A}$ on the open surface

$$S_\epsilon = \{\mathbf{x} : |\mathbf{x}| = 1,\ x^2 + y^2 \geq \epsilon^2\}$$

How does this help reconcile the existence of $\mathbf{A}$ with the result of Question 4?

6. Use Gauss' flux method to find the electric field $\mathbf{E} = \mathbf{E}(\mathbf{x})$ due to a spherically symmetric charge density

$$\rho(r) = \begin{cases} 0 & 0 \leq r \leq a \\ \rho_0 r/a & a < r < b \\ 0 & r \geq b \end{cases}$$

Now find the electric potential $\phi = \phi(r)$ *directly* from Poisson's equation by writing down the general, spherically symmetric solution to Laplace's equation in each of the intervals $0 < r < a$, $a < r < b$ and $r > b$, and adding a particular integral where necessary. You should assume that ϕ and ϕ' are continuous at $r = a$ and $r = b$. Check this solution gives rise to the same electric field using $\mathbf{E} = -\nabla\phi$.

7. The scalar field $\psi(r)$ only depends on $r = |\mathbf{x}|$. Use Cartesian coordinates and suffix notation to show

$$\nabla\psi = \psi'(r)\frac{\mathbf{x}}{r} \quad \text{and} \quad \nabla^2\psi = \psi''(r) + \frac{2}{r}\psi'(r).$$

Verify this result using your expression for the Laplacian in spherical polar coordinates. Find a non-singular, spherically symmetric solution to the equation $\nabla^2\psi = 1$ for $r < R$ subject to the requirement that $\psi(R) = 1$.

8. Consider a complex valued function $f = \phi(x, y) + i\psi(x, y)$, with ϕ and ψ real, satisfying $\partial f/\partial \bar{z} = 0$, where $\partial/\partial\bar{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$. Show that $\nabla^2\phi = \nabla^2\psi = 0$. Show also that a curve on which ϕ is constant is orthogonal to a curve on which ψ is constant, at a point where they intersect. Find ϕ and ψ when $f = ze^z$, $z = x + iy$, and compare with Question 5 on Examples Sheet 2.

9a. Using Cartesian coordinates (x, y), find all solutions of Laplace's equation $\nabla^2\psi = 0$ in two dimensions of the form $\psi(x, y) = f(x)e^{\alpha y}$, with α constant. Hence find a solution on the region $0 < x < a$ and $y > 0$ with boundary conditions:

$$\psi(0, y) = \psi(a, y) = 0 \quad \text{and} \quad \psi(x, 0) = \lambda \sin(\pi x/a)$$

and $\psi(x, y) \to 0$ as $y \to \infty$.

b. Using the formula for the 2d Laplacian in plane polar coordinates (r, θ), verify that Laplace's equation in the plane has solutions of the form $\psi(r, \theta) = Ar^\alpha \cos\beta\theta$, if α and β are related appropriately. Hence find solutions on the following regions, with the given boundary conditions (λ a constant):

(i) $r < R$ with $\psi(R, \theta) = \lambda \cos\theta$,

(ii) $r > R$ with $\psi(R, \theta) = \lambda \cos\theta$ and $\psi(r, \theta) \to 0$ as $r \to \infty$,

(iii) $a < r < b$ with $\mathbf{n} \cdot \nabla\psi(a, \theta) = 0$ and $\psi(b, \theta) = \lambda \cos 2\theta$.

10. Let ψ and ϕ be scalar functions. Using an integral theorem, establish *Green's second identity*

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) \mathrm{d}V = \int_{\partial V} (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathrm{d}\mathbf{S}$$

11. Show that if the following boundary value problem has a solution on V, then that solution is unique:

$$-\nabla^2\psi + \psi = \rho(\mathbf{x})$$

with $\mathbf{n} \cdot \nabla\psi = f(\mathbf{x})$ on ∂V.

12. Consider the Laplace equation $\nabla^2\psi = 0$ on V, subject to the boundary condition on ∂V

$$(\mathbf{n} \cdot \nabla\psi)g(\mathbf{x}) + \psi = f(\mathbf{x})$$

where $g(\mathbf{x}) \geq 0$ on ∂V. Show that, if a solution exists, then it is unique. Find a non-zero solution to Laplace's equation on $|\mathbf{x}| \leq 1$ which satisfies the boundary conditions above with $f = 0$ and $g = -1$ on $|\mathbf{x}| = 1$.

13. Let u be harmonic on V and v a smooth function that satisfies $v = 0$ on ∂V. Show that

$$\int_V \nabla u \cdot \nabla v \, \mathrm{d}V = 0.$$

Now if w is any function on V with $w = u$ on ∂V, show, by considering $v = w - u$, that

$$\int_V |\nabla w|^2 \, \mathrm{d}V \geq \int_V |\nabla u|^2 \, \mathrm{d}V.$$

14*. Show that a harmonic function ψ at the point $\mathbf{a}$ is equal to the average of its values on the interior of the ball $B_r(\mathbf{a}) = \{\mathbf{x} : |\mathbf{x} - \mathbf{a}| < r\}$, for any $r > 0$. Using this result for large r and considering $\nabla\psi$, or otherwise, prove that if ψ is bounded and harmonic on $\mathbb{R}^3$ then it is constant.

15*. Consider a time-dependent volume $V = V(t)$. The velocity of a point $\mathbf{x} \in V$ is $\mathbf{v}(\mathbf{x})$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(V) = \int_S \mathbf{v} \cdot \mathrm{d}\mathbf{S}.$$

Show that, for a scalar function $\rho(\mathbf{x}, t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \rho \, \mathrm{d}V = \int_{V(t)} \frac{\partial \rho}{\partial t} \, \mathrm{d}V + \int_{S(t)} \rho \mathbf{v} \cdot \mathrm{d}\mathbf{S} \ .$$

This is *Reynold's Transport Theorem*. What is the physical interpretation?

[Hint: it is better to think physically about this problem rather than simply trying to manipulate equations. You might first try constructing a 1d version of the result.]