Vector Calculus: Example Sheet 1

David Tong, January 2024

We will have covered the necessary material to do attempt all these questions by the end of lecture 7.

1. Sketch the curve in the plane given parametrically by

$$\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t), \quad 0 \le t \le 2\pi.$$

Calculate $\dot{\mathbf{x}}(t)$ at each point and hence find its total length.

2. A circular helix is given by

$$\mathbf{x}(t) = (a\cos t, a\sin t, ct), \quad t \in \mathbb{R}.$$

Calculate the tangent $\mathbf{t}$, curvature κ, principal normal $\mathbf{n}$, binormal $\mathbf{b}$ and torsion τ. Give a sketch of the curve indicating the directions of the vectors $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$.

3. Show that a planar curve $\mathbf{x}(t) = (x(t), y(t), 0)$ has curvature

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

Find the minimum and maximum values of the curvature on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

4. Evaluate explicitly each of the line integrals

$$\int (x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)\,, \quad \int (y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}z)\,, \quad \int \left(y\,\mathrm{d}x - x\,\mathrm{d}y + e^{x+y}\,\mathrm{d}z\right)$$

along (i) the straight line path from the origin to $(1,1,1)$, and (ii) the parabolic path given parametrically by $(x,y,z) = (t,t,t^2)$ from $t=0$ to $t=1$. For which of these integrals do the two paths give the same results, and why?

5. Consider the vector fields $\mathbf{F}(\mathbf{x}) = (3x^2yz^2, 2x^3yz, x^3z^2)$ and $\mathbf{G}(\mathbf{x}) = (3x^2y^2z, 2x^3yz, x^3y^2)$. Compute the line integrals $\int \mathbf{F}\cdot \mathrm{d}\mathbf{x}$ and $\int \mathbf{G}\cdot \mathrm{d}\mathbf{x}$ along the following paths, each of which consist of straight line segments joining the specified points:

(i) $(0,0,0) \to (1,1,1)$,

(ii) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$,

(iii) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$.

Are either of the differentials $\mathbf{F}\cdot \mathrm{d}\mathbf{x}$ or $\mathbf{G}\cdot \mathrm{d}\mathbf{x}$ exact?

6. A curve C is given parametrically by

$$\mathbf{x}(t) = (\cos(\sin nt)\cos t\ ,\ \cos(\sin nt)\sin t\ ,\ \sin(\sin nt)) \quad 0 \le t \le 2\pi,$$

where n is some fixed integer. Sketch the curve. Show that

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{x} = 2\pi, \quad \text{where} \quad \mathbf{F}(\mathbf{x}) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

and C is traversed in the direction of increasing t. Can $\mathbf{F}(\mathbf{x})$ be written as the gradient of a scalar function? Comment on your results.

[Hint: when sketching the curve, you may find it helpful to use spherical polar coordinates, defined by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$ with $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi)$.]

7. Use the substitution $x = r\cos\phi$, $y = \frac{1}{2}r\sin\phi$, to evaluate

$$\int_D \frac{x^2}{x^2+4y^2}\,\mathrm{d}A$$

where D is the region between the two ellipses $x^2+4y^2 = 1$, $x^2+4y^2 = 4$.

8. A closed curve C in the $z = 0$ plane consists of the arc of the parabola $y^2 = 4ax$ ($a > 0$) between the points $(a, \pm 2a)$ and the straight line joining $(a, \mp 2a)$. The area inclosed by C is D. Show, by calculating the integrals explicitly, that

$$\oint_C \left(x^2y\,\mathrm{d}x + xy^2\,\mathrm{d}y\right) = \int_D (y^2 - x^2)\,\mathrm{d}A = \frac{104}{105}a^4$$

9. The region D is bounded by the segments $x = 0$, $0 \le y \le 1; y = 0, 0 \le x \le 1; y = 1, 0 \le x \le \frac{3}{4}$, and by an arc of the parabola $y^2 = 4(1-x)$. Consider a mapping into the (x, y)-plane from the (u, v)-plane defined by the transformation $x = u^2 - v^2$, $y = 2uv$. Sketch D and also the two regions in the (u, v)-plane which are mapped into it. Hence evaluate

$$\int_D \frac{\mathrm{d}A}{(x^2 + y^2)^{1/2}}$$

10. Compute the volume of a cone of height h and radius a using (a) cylindrical polars, (b) spherical polars.

11. By using a suitable change of variables, calculate the volume within an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1.$$

12*. Compute the volume of the region V defined by the intersection of the three cylinders

$$V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le 1,\ y^2 + z^2 \le 1,\ z^2 + x^2 \le 1\}.$$

[Warning: The sole purpose of this question is to show you that volume integrals can be arbitrarily hard. Only attempt if that's your thing.]