Please attempt questions 1–10.

Some questions may be unfamiliar in style and on some you might get stuck. This is okay! If you can't do a question, don't panic – write down your ideas, then we can discuss them and work towards a solution in the supervision. However, please don't look up any solutions. I can help you if I get to see your attempts, but it's fairly useless if I just get to see someone else's answer!

You are welcome to mail me for help (glt1000@cam.ac.uk), but you should tell me what you've tried.

1. By considering $(r+1)^3 - r^3$, derive the formula $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

2. Use induction to prove that, for every positive integer n,

   (i) $1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{1}{3}(4n^3 - n)$

   (ii) $n^3 + 5n$ is divisible by 6

   (iii) $2^{n+2} + 3^{2n+1}$ is divisible by 7.

3. Give alternative solutions to question 2, as follows:

   (i) for 2(i), by using the formula in question 1

   (ii) for 2(ii), by factorising a suitable expression

   (iii) for 2(iii), by using modular arithmetic (or, if you haven't met modular arithmetic, using the result that $a - b$ divides $a^k - b^k$ for $a, b \in \mathbb{Z}$ and $k \in \mathbb{N}$).

4. *Theorem.* All cows are the same colour.

   Proof. We will use induction to show that, for each n, any n cows are the same colour. The base case, $n = 1$, is easy: any one cow is the same colour as itself. Now suppose the result is true for any set of n cows, and that we have $n+1$ cows. Call them $c_1, \ldots, c_{n+1}$.

   By induction, we know cows $c_1, \ldots, c_n$ are the same colour (because there are n of them), and that cows $c_2, \ldots, c_{n+1}$ are the same colour (because there are n of them). So we deduce that cows $c_1, \ldots, c_{n+1}$ are the same colour. Hence, by induction, all cows are the same colour. $\square$

   However, not all cows are the same colour. So where is the mistake?

5. The Fibonacci numbers are defined by: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$.

   (a) Let n be a positive integer. By observing that

   $$F_{n+2} = F_{n+1} + F_n, \quad F_{n+3} = 2F_{n+1} + F_n, \quad F_{n+4} = 3F_{n+1} + 2F_n, \quad \ldots,$$

   guess a formula for F_{n+k} in terms of F_{n+1} and F_n, and verify it by induction on k.

   Deduce that $F_{n+1}^2 + F_n^2$ and $F_{n+2}^2 - F_n^2$ are Fibonacci numbers.

   Deduce also that F_{kn} is a multiple of F_n for all $k \in \mathbb{N}$, and hence show that if F_n is prime then either n is prime or $n = 4$.

   (b) For each n, let f_n be the last digit of F_n. For example, $F_7 = 13$, so $f_7 = 3$. Prove that the sequence f_n is periodic. (That is, prove that there is some positive integer k such that $f_{n+k} = f_n$ for all n.)

6. (a) Find all positive integers n such that $n!$ is the difference of two squares.
   (b) Find all positive integers n such that n divides $(n-1)!$.
   (c) Find the smallest positive integer n such that $n!$ ends in (at least) 2021 zeroes.

7. (a) Prove that $\sqrt[3]{4}$ and $\log_3 4$ are irrational.
   (b) Show that if there are $m, n \in \mathbb{N}$ such that $\dfrac{m}{n} = \sqrt{11}$, then also $\dfrac{11n - 3m}{m - 3n} = \sqrt{11}$.

   By considering the size of $m - 3n$, deduce that $\sqrt{11}$ is irrational.

   Can you generalise this method? E.g., what fraction could we use for $\sqrt{111}$, and at what step does the method fail for $\sqrt{121}$?

   (c) Let $r, \alpha, \beta \in \mathbb{R}$, with r rational and α, β irrational. Which, if any, of $r + \alpha$, $r\alpha$, $\alpha + \beta$, $\alpha\beta$, r^α, α^r and α^β must be irrational? Give proofs or counterexamples.

   If you give a counterexample, try to use α, β that you can prove are irrational.

8. Let S be a set of $n+1$ distinct integers chosen from $\{1, \ldots, 2n\}$. Prove that S contains:
   (i) two numbers which are coprime
   (ii) two numbers whose sum is $2n+1$
   (iii) two numbers such that one divides the other
   (iv) two numbers whose difference is also in S.

   Give examples (with $n > 1$) to show that each result can fail if S contains only n integers. Is there such an example where all four results fail simultaneously?

9. You are asked to drive a lunar rover around the moon (which is just a circle in this question). There are some fuel depots on the way, with the total amount of fuel stored in them enough to get around the moon exactly once. Prove that there exists a depot from which you can start driving and travel the whole way around the moon, picking up fuel at each depot as you pass, without running out of fuel between depots.

10. There are six towns, such that between each pair of towns there is either a train or bus service (but not both). Prove that there are three towns that can be visited in a loop, going via no other towns, using only one mode of transport.

    Is the result still true if there are only five towns?

Additional questions

These are optional. Attempt them if they interest you, but not at the expense of other work.

11. The region in question 10 has grown and there are now eighteen towns. As before, between each pair of towns there is either a train or bus service (but not both). Prove that there are four towns such that all six of their pairwise connections use the same mode of transport.

12. A natural number is a *triangular number* if it is of the form $\frac{1}{2}n(n+1)$ for some $n \in \mathbb{N}$. Find all natural numbers that cannot be written as a sum of distinct triangular numbers.

13. Let R be a rectangle which can be divided into smaller rectangles, each of which has at least one side of integer length. Prove that R has at least one side of integer length.