

IA Numbers & Sets – Example Sheet 1

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Questions marked † are more challenging.

1. The numbers 3, 5, 7 are all prime. Does it ever happen again that three numbers of the form $n, n + 2, n + 4$ are all prime?
2. There are four primes between 0 and 10 and between 10 and 20. Does it ever happen again that there are four primes between two consecutive multiples of 10?
3. Consider the sequence 41, 43, 47, 53, 61, ... (where each difference is 2 more than the previous one). Are all of these numbers prime?
4. Translate the following sentence into a short English one. Is it true? Write down its negation in symbolic form. (Here m, n, a, b should be understood as ranging over all natural numbers.)

$$\forall m \exists n \forall a \forall b (n \geq m) \wedge [(a = 1) \vee (b = 1) \vee (ab \neq n)]$$

5. Find a prime factor of $2^{19} + 5^{40}$. Can you find another? Show that $2^{91} - 1$ is not prime.
6. If n^2 is a multiple of 3, must n be a multiple of 3?
7. Use induction to show that for every positive integer n the number $2^{n+2} + 3^{2n+1}$ is a multiple of 7.
8. Write down carefully (while not looking at your notes) a proof that there are infinitely many primes. By considering numbers of the form $4p_1p_2 \dots p_k - 1$, prove that there are infinitely many primes of the form $4n - 1$. What would go wrong if we tried a similar proof to show that there are infinitely many primes of the form $4n + 1$?
9. Prove that $2^{2^n} - 1$ has at least n distinct prime factors.
10. We are given an operation $*$ on the positive integers, satisfying
 - (i) $1 * n = n + 1$ for all n ;
 - (ii) $m * 1 = (m - 1) * 2$ for all $m > 1$;
 - (iii) $m * n = (m - 1) * (m * (n - 1))$ for all $m, n > 1$.

Find the value of $5 * 5$.

11. Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be?
12. Let $a < b$ be distinct natural numbers. Prove that every block of b consecutive natural numbers contains two distinct numbers whose product is a multiple of ab .

† Suppose now that $a < b < c$. Given an example of a block of c consecutive natural numbers such that no three distinct numbers in the block have a product that is a multiple of abc .
13. *Fun bonus question:* All integers greater than one but less than 100 are put into a hat and two are drawn. Sabrina is given their sum and Peter their product. Peter says, “I can tell you don’t know the numbers.” Peter replies, “Now I do.” Sabrina exclaims, “Now I do too!” What are the numbers?
14. *Fun bonus question:* Each card below has a number on one side and colour on the other. Which card(s) must be turned over to test the idea that if a card shows an even number on one face, then its opposite face is blue?

