IA Numbers & Sets – Example Sheet 4

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Questions marked † are more challenging.

1. Which of the following sequences $(x_n)_{n=1}^{\infty}$ converge?

   (i) $x_n = \frac{3n}{n+3}$ (ii) $x_n = \frac{n^{100}}{2^n}$ (iii) $x_n = \sqrt{n+1} - \sqrt{n}$ (iv) $x_n = (n!)^{1/n}$

2. Define a sequence $(x_n)_{n=1}^{\infty}$ by setting $x_1 = 1$ and $x_{n+1} = \frac{x_n}{1+\sqrt{x_n}}$ for all $n \geq 1$. Show that $(x_n)_{n=1}^{\infty}$ converges, and determine its limit.

3. Which of the following series converge?

   (i) $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ (ii) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (iii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$

4. Let $\sum_{n=1}^{\infty} x_n$ be a convergent series of positive reals.

   (i) Must $\sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$ be convergent?

   (ii) Show that $\sum_{n=1}^{\infty} x_n^2$ is convergent. What happens if we do not insist that the x_n are positive?

5. Let $\sum_{n=1}^{\infty} x_n$ be a convergent series of reals. Must $\sum_{n=1}^{\infty} x_n^3$ be convergent?

6. A real number $x = 0.x_1x_2x_3\ldots$ is called *repetitive* if its decimal expansion contains arbitrarily long blocks that are the same, i.e. if for every k there exist distinct m and n such that $x_m = x_n,\ x_{m+1} = x_{n+1},\ \ldots,\ x_{m+k} = x_{n+k}$. Prove that the square of a repetitive number is repetitive.

7. Find an injection from $\mathbb{R}^2$ to $\mathbb{R}$. Is there an injection from the set of all real sequences to $\mathbb{R}$?

8. Let F be the collection of all finite subsets of $\mathbb{N}$. Is F countable?

9. Show that there does not exist an uncountable family of pairwise disjoint discs in the plane. What happens if we replace 'discs' by 'circles'?

10. A function $f : \mathbb{N} \to \mathbb{N}$ is *increasing* if $f(n+1) \geq f(n)$ for all n and *decreasing* if $f(n+1) \leq f(n)$ for all n. Is the set of increasing functions countable or uncountable? What about the set of decreasing functions?

11. Let S be a collection of subsets of $\mathbb{N}$ such that for every $A, B \in S$ we have $A \subset B$ or $B \subset A$. Can S be uncountable? Is there an uncountable collection T of subsets of $\mathbb{N}$ such that $A \cap B$ is finite for all distinct $A, B \in T$?

12. Find a bijection from the rationals to the non-zero rationals. Is there such a bijection that is order-preserving (i.e. $x < y$ implies $f(x) < f(y)$)?

13. † In an unusual approach to Secret Santa, the Numbers and Sets lecturer places, for each of the 260 IA students, a gift labelled with the student's name in one of 260 separate but identical boxes. On the final day of term, the boxes are lined up in the Babbage. One by one, students are invited to enter the room and inspect the content of up to 130 of the boxes. They must then leave the room and are permitted no further communication with each other. Students will only receive their gifts if every single student finds their own gift through this process.

    Students have a chance to plot their strategy in advance. Is there one that succeeds with probability at least 30%?