IA Numbers & Sets – Example Sheet 2

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Questions marked † are more challenging.

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1. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$? How many functions $A \to B$ are there? How many are injections? How many surjections from $B \to A$?

2. How many subsets of $\{1, 2, 3, 4\}$ have even size? Based on your answer, guess and prove a formula for the number of subsets of $\{1, 2, \ldots, n\}$ of even size.

3. By suitably interpreting each side, establish the identities
$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} = \binom{n+1}{k+1}$$
and
$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n},$$
for appropriate ranges of the parameters n and k (which you should specify).

4. Let $r(n)$ denote the number of equivalence relations on a set with n elements. Show that $2^{n-1} \leq r(n) \leq 2^{n(n-1)/2}$.

5. Write down carefully (while not looking at your notes) a proof that there are infinitely many primes. By considering numbers of the form $4p_1p_2 \ldots p_k - 1$, prove that there are infinitely many primes of the form $4n - 1$. What would go wrong if we tried a similar proof to show that there are infinitely many primes of the form $4n + 1$?

6. Prove that $2^{2^n} - 1$ has at least n distinct prime factors.

7. Find integers x and y with $76x+45y = 1$. Do there exist integers x and y with $3381x+2646y = 21$?

8. Let $a, b, c, d \in \mathbb{N}$. Must the numbers $(a, b)(c, d)$ and (ac, bd) be equal? If not, must one be a factor of the other? If $(a, b) = (a, c) = 1$, must we have $(a, bc) = 1$?

9. Use the inclusion-exclusion principle to count the number of positive integers up to 1001 that are coprime to 1001.

10. The *repeat* of a positive integer is obtained by writing it twice in a row. For example, the repeat of 254 is 254254. Is there a positive integer whose repeat is a square number?

11. The *Fibonacci numbers* $F_1, F_2, F_3, \ldots$ are defined by: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$ (so eg. $F_3 = 2$, $F_4 = 3$, $F_5 = 5$). Is F_{2024} even or odd? Is it a multiple of 3?

12. Show that $2^{19} + 5^{40}$ is not prime. Show also that $2^{91} - 1$ is not prime.

13. Show that a positive integer n is a multiple of 9 if and only if the sum of its digits is a multiple of 9. The number 2^{29} has nine distinct digits. Which digit is missing?

14. Does there exist a block of 100 consecutive positive integers, none of which is prime?

15. Let $a < b$ be distinct natural numbers. Prove that every block of b consecutive natural numbers contains two distinct numbers whose product is a multiple of ab.

    † Suppose now that $a < b < c$. Must every block of c consecutive natural numbers contain three distinct numbers whose product is a multiple of abc?