IA Numbers & Sets – Example Sheet 1

Michaelmas 2025 zoe.wyatt@maths.cam.ac.uk

Questions marked † are more challenging.

1. If n^2 is a multiple of 3, must n be a multiple of 3?

2. Consider the sequence $41, 43, 47, 53, 61, \ldots$ (where each difference is 2 more than the previous one). Are all of these numbers prime?

3. There are four primes between 0 and 10, and between 10 and 20. Does it ever happen again that there are four primes between two consecutive multiples of 10?

4. Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be?

5. Write down the negation of the following assertions (where $m, n, a, b \in \mathbb{N}$):

   (i) $\forall m \; \exists n \; \forall a \; \forall b \; (n \geq m) \wedge [(a = 1) \vee (b = 1) \vee (ab \neq n)]$

   (ii) if Bumrah is not a faster bowler than Tait, then Australia is worse than England in cricket.

6. Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

7. Let $A_1, A_2, \ldots$ be sets such that for each $n \in \mathbb{N}$, we have $A_1 \cap \ldots \cap A_n \; \neq \; \emptyset$.

   Can we have $A_1 \cap A_2 \cap \ldots \; = \; \emptyset$?

8. Does $f \circ g$ injective imply f injective? Does it imply g injective? What happens if we replace 'injective' by 'surjective'?

9. Let $f : X \to Y$. Let $A, B \subset X$ and $C, D \subset Y$. For each of the claims below, give a proof or counterexample:

$$\begin{array}{llll} (i) & f(A \cup B) = f(A) \cup f(B) & (ii) & f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D) \\ (iii) & f(A \cap B) = f(A) \cap f(B) & (iv) & f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D) \\ (v) & f^{-1}(f(A)) = A & (vi) & f(f^{-1}(C)) = C \end{array}$$

   Show that each false claim can be made true by replacing " $=$ " by either " $\subset$ " or " $\supset$ "

10. Define a relation R on $\mathbb{N}$ by setting aRb if a divides b or b divides a. Is R an equivalence relation?

11. The relation S contains the relation R if aSb whenever aRb. Let R be the relation on $\mathbb{Z}$ given by aRb if $b = a + 3$. How many equivalence relations on $\mathbb{Z}$ contain R?

12. We are given an operation $*$ on the positive integers, satisfying

    (i) $1 * n = n + 1$ for all n;

    (ii) $m * 1 = (m - 1) * 2$ for all $m > 1$;

    (iii) $m * n = (m - 1) * (m * (n - 1))$ for all $m, n > 1$.

    Find the value of $5 * 5$.

13. The *symmetric difference* $A \triangle B$ of two sets A and B is the set of elements that belong to exactly one of A and B. Express this in terms of $\cap, \cup$ and $\setminus$. Prove that $\triangle$ is associative.

14. † Each of n sprightly freshers knows a piece of information not known to any of the others. They communicate by Snapchat, and in each video call the two freshers concerned reveal to each other all the information they know so far. What is the smallest number of calls that can be made in such a way that, at the end, all the freshers know all the information?

15. *Bonus question to ask your friends in the college bar:* each card below has a number on one side and color on the other. Which card(s) must be turned over to test the idea that if a card shows an even number on one face, then its opposite face is blue?