IA Numbers & Sets – Example Sheet 3

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Questions marked † are more challenging.

1. Prove carefully, using the least upper bound axiom, that there is a real number x satisfying $x^3 = 2$. Prove also that such an x must be irrational.

2. Prove that $\sqrt{2} + \sqrt{3}$ is irrational and algebraic.

3. Suppose that $x \in \mathbb{R}$ is a root of a monic integer polynomial, i.e. we have $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 = 0$, for some integers $a_{n-1}, \ldots, a_0$. Prove that x is either integer or irrational.

4. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences of reals. Show that if $x_n \to 0$ and $y_n \to 0$, then $x_n y_n \to 0$. By considering $x_n - c$ and $y_n - d$, prove carefully that if $x_n \to c$ and $y_n \to d$, then $x_n y_n \to cd$.

5. Let $(x_n)_{n=1}^{\infty}$ be a sequence of reals. Show that if $(x_n)_{n=1}^{\infty}$ is convergent, then we must have $x_n - x_{n-1} \to 0$. If $x_n - x_{n-1} \to 0$, must $(x_n)_{n=1}^{\infty}$ be convergent?

6. Which of the following sequences $(x_n)_{n=1}^{\infty}$ converge?
   (i) $x_n = \frac{3n}{n+3}$ (ii) $x_n = \frac{n^{100}}{2^n}$ (iii) $x_n = \sqrt{n+1} - \sqrt{n}$ (iv) $x_n = (n!)^{1/n}$

7. Which of the following series converge?
   (i) $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ (ii) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (iii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$

8. Define a sequence $(x_n)_{n=1}^{\infty}$ by setting $x_1 = 1$ and $x_{n+1} = \frac{x_n}{1+\sqrt{x_n}}$ for all $n \geq 1$. Show that $(x_n)_{n=1}^{\infty}$ converges, and determine its limit.

9. A real number $x = 0.x_1x_2x_3\ldots$ is called *repetitive* if its decimal expansion contains arbitrarily long blocks that are the same, i.e. if for every k there exist distinct m and n such that $x_m = x_n,\ x_{m+1} = x_{n+1},\ \ldots,\ x_{m+k} = x_{n+k}$. Prove that the square of a repetitive number is repetitive.

10. Show that if $\sum_{n=1}^{\infty} x_n$ is a convergent series of reals, with all x_n positive, then $\sum_{n=1}^{\infty} x_n^2$ is also convergent. What happens if we do not insist that the x_n are positive?

11. If $\sum_{n=1}^{\infty} x_n$ is a convergent series of reals, must $\sum_{n=1}^{\infty} x_n^3$ be convergent?

12. Show that $\sqrt[100]{\sqrt{3} + \sqrt{2}} + \sqrt[100]{\sqrt{3} - \sqrt{2}}$ is irrational.

13. If $\sum_{n=1}^{\infty} x_n$ is a convergent series of reals, must $\sum_{n=1}^{\infty} \frac{x_n}{n}$ be convergent?

14. † Let S be a (possibly infinite) set of odd positive integers. Prove that there exists a real sequence $(x_n)_{n=1}^{\infty}$ such that, for every positive integer k, the series $\sum_{n=1}^{n} x_n^k$ converges when k belongs to S and diverges when k does not belong to S.