IA Numbers & Sets – Example Sheet 2

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Questions marked † are more challenging.

1. Find integers x and y with $76x+45y=1$. Do there exist integers x and y with $3381x+2646y=21$?
2. Let $a,b,c,d \in \mathbb{N}$. Must the numbers $(a,b)(c,d)$ and (ac,bd) be equal? If not, must one be a factor of the other? If $(a,b)=(a,c)=1$, must we have $(a,bc)=1$?
3. Show that a positive integer n is a multiple of 9 if and only if the sum of its digits is a multiple of 9. The number 2^{29} has nine distinct digits. Which digit is missing?
4. The *Fibonacci numbers* $F_1, F_2, F_3, \dots$ are defined by: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$ (so eg. $F_3 = 2$, $F_4 = 3$, $F_5 = 5$). Is F_{2024} even or odd? Is it a multiple of 3?
5. Solve (i.e. find all solutions of) the equations
   (i) $7x \equiv 77 \bmod 40$;
   (ii) $12y \equiv 30 \bmod 54$;
   (iii) $3z \equiv 2 \bmod 17$ and $4z \equiv 3 \bmod 19$.
6. An RSA encryption scheme (n,e) has modulus $n=187$ and encoding exponent $e=7$. Find a suitable decoding exponent d. Check your answer (without electronic assistance) by explicitly encoding the number 35 and then decoding the result.
7. By considering the n fractions $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}$ or otherwise, prove that $n = \sum_{d|n} \varphi(d)$.
8. Explain (without electronic assistance) why 23 cannot divide $10^{881}-1$.
9. Let p be a prime of the form $3k+2$. Show that if $x^3 \equiv 1 \bmod p$, then $x \equiv 1 \bmod p$. Deduce that every number is a cube modulo p, that is, $y^3 \equiv a \bmod p$ has an integer solution y for all $a \in \mathbb{Z}$.
10. By considering numbers of the form $(2p_1p_2\dots p_k)^2+1$, prove that there are infinitely many primes of the form $4n+1$.
11. What is the 5th-last digit of $5^{5^{5^{5^5}}}$?
12. Is there a positive integer n for which $n^7 - 77$ is a Fibonacci number?
13. Let A be the sum of the digits of 4444^{4444}, and let B be the sum of the digits of A. What is the sum of the digits of B?
14. † Let n and k be positive integers. Suppose that n is a kth power modulo p for all primes p. Must n be a kth power?