Michaelmas Term 2023-24

Numbers and Sets: Examples Sheet 4 of 4

1. How many subsets of $\{1, 2, 3, 4\}$ have even size? Based on your answer, guess and prove a formula for the number of subsets of $\{1, 2, \ldots, n\}$ of even size.

2. By suitably interpreting each side, establish the identities

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} = \binom{n+1}{k+1}$$

and

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n},$$

for appropriate ranges of the parameters n and k (which you should specify).

3. The *symmetric difference* of two sets A and B is $A \bigtriangleup B = (A \setminus B) \cup (B \setminus A)$. Give a direct proof that the operation $\bigtriangleup$ is associative, and then another one based on indicator functions mod 2.

4. Use the inclusion-exclusion principle to determine $\phi(1001)$.

5. Let $A_1, A_2, \ldots$ be sets such that for each $n \in \mathbb{N}$, we have $A_1 \cap \ldots \cap A_n \neq \emptyset$. Can we have $A_1 \cap A_2 \cap \ldots = \emptyset$?

6. Does $f \circ g$ injective imply f injective? Does it imply g injective? What happens if we replace 'injective' by 'surjective'?

7. Find an injection from $\mathbb{R}^2$ to $\mathbb{R}$. Is there an injection from the set of all real sequences to $\mathbb{R}$?

8. Define a relation R on $\mathbb{N}$ by setting aRb if a divides b or b divides a. Is R an equivalence relation?

9. Let $r(n)$ denote the number of equivalence relations on a set with n elements. Show that $2^{n-1} \leq r(n) \leq 2^{n(n-1)/2}$. Can you give a stronger upper bound?

10. Show that there does not exist an uncountable family of pairwise disjoint discs in the plane. What happens if we replace 'discs' by 'circles'?

11. Let F be the collection of all finite subsets of $\mathbb{N}$. Is F countable?

12. A function $f : \mathbb{N} \to \mathbb{N}$ is *increasing* if $f(n+1) \geq f(n)$ for all n and *decreasing* if $f(n+1) \leq f(n)$ for all n. Is the set of increasing functions countable or uncountable? What about the set of decreasing functions?

13. Let S be a collection of subsets of $\mathbb{N}$ such that for every $A, B \in S$ we have $A \subset B$ or $B \subset A$. Can S be uncountable? Is there an uncountable collection T of subsets of $\mathbb{N}$ such that $A \cap B$ is finite for all distinct $A, B \in T$?

14. Find a bijection from the rationals to the non-zero rationals. Is there such a bijection that is order-preserving (i.e. $x < y$ implies $f(x) < f(y)$)?

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+15. In an unusual approach to Secret Santa, the Numbers and Sets lecturer places, for each of the 260 IA students, a gift labelled with the student's name in one of 260 separate but identical boxes. On the final day of term, the boxes are lined up in the Babbage. One by one, students are invited to enter the room and inspect the content of up to 130 of the boxes. They must then leave the room and are permitted no further communication with each other. Students will only receive their gifts if every single student finds their own gift through this process.

Students have a chance to plot their strategy in advance. Is there one that succeeds with probability at least 30%?