1. (a) Let (X_1, d_1) and (X_2, d_2) be metric spaces. Show that we may define a metric d on the product $X_1 \times X_2$ by $d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$.

   Show that the projections $\pi_i : X_1 \times X_2 \to X_i$, $(x_1, x_2) \mapsto x_i$, are continuous.

   Show that if (X_1, d_1) and (X_2, d_2) are complete, then so is $(X_1 \times X_2, d)$.

   (b) Let (X_i, d_i) be metric spaces for $i = 1, 2, \ldots$, and let X be the set of all sequences $(x_i)_{i=1}^{\infty}$ with $x_i \in X_i$ for all i. Show that we may define a metric d on X by

$$d\big((x_i), (y_i)\big) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} .$$

2. (a) Let d_1, d_2, d_∞ be the metrics on $\mathbb{R}^n$ given by

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|, \quad d_2(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}, \quad d_\infty(\mathbf{x}, \mathbf{y}) = \sup_i |x_i - y_i|.$$

   For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, show that

$$d_1(\mathbf{x}, \mathbf{y}) \geqslant d_2(\mathbf{x}, \mathbf{y}) \geqslant d_\infty(\mathbf{x}, \mathbf{y}) \geqslant \frac{1}{\sqrt{n}} d_2(\mathbf{x}, \mathbf{y}) \geqslant \frac{1}{n} d_1(\mathbf{x}, \mathbf{y}) .$$

   Deduce that the metrics induce the same topology on $\mathbb{R}^n$.

   (b) Now let d_1, d_2, d_∞ be the metrics on $C[0, 1]$ given by

$$d_1(f, g) = \int_0^1 |f - g|, \quad d_2(f, g) = \left[\int_0^1 (f - g)^2\right]^{1/2}, \quad d_\infty(f, g) = \sup_{[0,1]} |f - g|.$$

   Show that the metrics induce distinct topologies on $C[0, 1]$.

3. Define the maps $f, g : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x + y$ and $g(x, y) = xy$. Show that f, g are continuous with respect to the Euclidean topologies on $\mathbb{R}^2$ and $\mathbb{R}$.

   Now give $\mathbb{R}$ the topology τ in which the open sets are intervals of the form (a, ∞), and give $\mathbb{R}^2$ the resulting product topology. Are f, g continuous with respect to these topologies?

   Find all continuous functions from $(\mathbb{R}, \tau)$ to $(\mathbb{R}, \text{Euclidean})$.

4. Determine whether the following subsets of $\mathbb{R}^2$ are open, closed, both or neither.

   (i) $\{(x, y) \mid x < 0\} \cup \{(x, y) \mid x > 0, y > 1/x\}$

   (ii) $\{(x, \sin(1/x)) \mid x > 0\} \cup \{(0, y) \mid y \in [-1, 1]\}$

   (iii) $\{(x, y) \mid y = x^n \text{ for some positive integer } n\}$.

5. Show that $\mathbb{Q}$ is not complete with respect to the Euclidean metric.

   Is there a metric on $\mathbb{Q}$ which makes it into a complete metric space?

6. For a function $f : X \to Y$, define its *graph* to be $\Gamma_f = \{(x, f(x)) : x \in X\} \subseteq X \times Y$.

   Prove that $f : [0, 1] \to [0, 1]$ is continuous if and only if Γ_f is closed in $[0, 1]^2$.

   Give an example of $f : \mathbb{R} \to \mathbb{R}$ for which Γ_f is closed in $\mathbb{R}^2$ but f is not continuous.

7. Let (X, d) be a metric space. For $A \subseteq X$, define $d_A : X \to \mathbb{R}$ by $d_A(x) = \inf_{y \in A} d(x, y)$. Show that d_A is continuous, and that A is closed if and only if $d_A(x) > 0$ for all $x \notin A$.

   Let A, B be disjoint closed subsets of X. Show that there exist disjoint open subsets U, V of X with $A \subseteq U$ and $B \subseteq V$. Must we have $\inf_{x \in B} d_A(x) > 0$?

8. Let $f : X \to Y$ be a map of topological spaces. Show that f is continuous if and only if $f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$ for all $A \subseteq X$.

   Deduce that if f is continuous and surjective, then the image of a dense set in X is dense in Y.

9. Let X be a topological space. Show that the following statements are equivalent:

   (i) X is Hausdorff

   (ii) The diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$, with the product topology

   (iii) For any topological space Y and any continuous maps $f, g : Y \to X$, the set $\{y \in Y : f(y) = g(y)\}$ is closed in Y.

   Deduce that if X is Hausdorff and $f : Y \to X$ is a continuous function on a space Y, then f is determined by its values on any dense subset of Y.

   If instead the diagonal Δ is an open subset of $X \times X$, what is the topology on X?

10. A topological space is called *separable* if it has a countable dense subset, and is called *second countable* if it has a countable base of open sets.

    (a) Show that $\mathbb{R}$ with the Euclidean topology is separable and second countable.

    (b) Let X be $\mathbb{R}$ with the topology in which a subset of $\mathbb{R}$ is open if either it is empty or contains 0. Is X separable? Is X second countable?

    (c) Prove that a second countable topological space is separable, and that a separable metric space is second countable. Deduce that a subspace of a separable metric space is separable.

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11. Refer back to question 1(b). Prove that if each (X_i, d_i) is complete then so is (X, d), and that if each (X_i, d_i) is separable then so is (X, d).

12. Let X be $\mathbb{R}$ with the *cocountable topology*, in which a subset of $\mathbb{R}$ is open if either it is empty or its complement in $\mathbb{R}$ is countable. Is X separable? Is X second countable? Which sequences $(x_i)_{i=1}^{\infty}$ in X converge, and what can you say about the limit?

    Repeat with the *cofinite topology*, in which a subset of $\mathbb{R}$ is open if either it is empty or its complement in $\mathbb{R}$ is finite.

13. (a) Find a sequence $[a_1, b_1], [a_2, b_2], \ldots$ of closed intervals in $\mathbb{R}$ of positive length whose union contains all rationals in $[0, 1]$ and such that $\sum_{i=1}^{\infty}(b_i - a_i) < 1$.

    (b) Let $[a_1, b_1], [a_2, b_2], \ldots$ be a sequence of closed intervals in $\mathbb{R}$ of positive length whose union contains all irrationals in $[0, 1]$. Can we have $\sum_{i=1}^{\infty}(b_i - a_i) < 1$?

E. 2023 **METRIC AND TOPOLOGICAL SPACES – SHEET 2** G. Taylor

1. Let $\mathbb{R}$ have the Euclidean topology, and let $\sim$ be the equivalence relation defined by $x \sim y$ if and only if either $x = y = 0$ or $xy > 0$. Find all open sets of the quotient space $\mathbb{R}/\sim$. Is $\mathbb{R}/\sim$ Hausdorff?

   Now let $\sim$ instead be the equivalence relation on $\mathbb{R}$ defined by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. What is the topology on $\mathbb{R}/\sim$?

2. Determine whether the following subsets of $\mathbb{R}^2$ are connected, path-connected, both or neither. Here, $B_r(\mathbf{x})$ is the open ball with centre $\mathbf{x}$ and radius r, and $\overline{B}_r(\mathbf{x})$ is the corresponding closed ball.

   (i) $B_1((1,0)) \cup B_1((-1,0))$
   (ii) $B_1((1,0)) \cup \overline{B}_1((-1,0))$
   (iii) $\{(x,y) \mid \text{at least one of } x \text{ and } y \text{ is rational}\}$
   (iv) $\{(x,y) \mid \text{exactly one of } x \text{ and } y \text{ is rational}\}$.

3. Let X be a connected topological space.

   (a) Let $f : X \to Y$ be a locally constant map to a topological space Y, i.e. for every $x \in X$, there is an open neighbourhood of x on which f is constant. Show that f is constant.

   (b) Suppose that for every $x \in X$, there is an open neighbourhood of x which is path-connected. Show that X is path-connected.

4. Let X be a topological space.

   (a) Let A_i, $i \in I$, be a collection of connected subsets of X such that $A_i \cap A_j \neq \emptyset$ for all $i, j \in I$. Prove that $\bigcup_{i \in I} A_i$ is connected.

   (b) Prove that if A is a connected subset of X, then $\mathrm{cl}(A)$ is also connected. Deduce that any connected component of X is closed.

   (c) Prove that every connected component of the product $X \times X$ is a product of connected components of X.

5. (a) Is there a metric on $\mathbb{Q}$ which makes it into a connected space? What about $\mathbb{R}\backslash\mathbb{Q}$?

   (b) Is there an infinite compact subset of $\mathbb{Q}$, in the Euclidean topology?

6. I am standing in a forest on $\mathbb{R}^2$ and cannot see anything but trees in every direction. Is it possible to cut down all but finitely many trees so that I still can't see out?

7. A collection $\mathcal{C}$ of subsets of a topological space is said to have the *finite intersection property* if every finite subcollection of $\mathcal{C}$ has non-empty intersection.

   Prove that a topological space is compact if and only if, for every collection of closed subsets with the finite intersection property, the whole collection has non-empty intersection.

8. (a) Show that a subset A of $\mathbb{R}^n$ is compact if and only if every continuous function from A to $\mathbb{R}$ has bounded image.

   (b) Show that any open cover of $\mathbb{R}^n$ has a countable subcover.

9. Let X be a topological space. Its *one-point compactification* X^* is defined as follows. As a set, X^* is the union of X with an additional point denoted by ∞. A subset U of X^* is open if either

   (i) $\infty \notin U$ and U is open in X, or
   (ii) $\infty \in U$ and $X^* \setminus U$ is a closed and compact subset of X.

   Show that this defines a topology, and that X^* is compact. When is X dense in X^*?

   Write down a subset of $\mathbb{R}$ which is homeomorphic to $\mathbb{Z}^*$.

   Show that $\mathbb{C}^*$ is homeomorphic to the sphere S^2.

10. Let $\mathbb{T}^2$ be the two-dimensional torus, defined as $\mathbb{R}^2/\sim$, where $(x, y) \sim (x', y')$ if and only if $x - x'$ and $y - y'$ are both integers.

    (a) Show that $\mathbb{T}^2$ is compact and path-connected.

    (b) Let $L \subset \mathbb{R}^2$ be a line of the form $y = \alpha x$, where α is irrational, and let $q(L)$ be its image in $\mathbb{T}^2$, where $q : \mathbb{R}^2 \to \mathbb{T}^2$ is the quotient map.

    What are the interior and closure of $q(L)$ in $\mathbb{T}^2$?

    Show that the restriction of q to L is a continuous bijection from L to $q(L)$. Is it a homeomorphism?

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11. Refer back to question 1(b) on sheet 1. Prove that if each (X_i, d_i) is connected then so is (X, d), and that if each (X_i, d_i) is compact then so is (X, d).

12. Can a topological space be homeomorphic to its own one-point compactification?

13. For each of the spaces X below, is the one-point compactification X^* metrizable?

    (i) $X = \mathbb{R}$, with the Euclidean topology
    (ii) $X = \mathbb{R}$, with the discrete topology
    (iii) $X = \mathbb{Q}$, with the Euclidean topology
    (iv) $X = \mathbb{Q}$, with the discrete topology.

1. Let $\mathbb{Z}$ have the 5-adic metric. Show that:

   (i) the sequence 2020, 20020, 200020, 2000020, ... converges

   (ii) the sequence 2020, 22020, 222020, 2222020, ... is Cauchy but doesn't converge.

2. (a) Let $A \subseteq \mathbb{R}^2$ be a countable set of points. Show that $\mathbb{R}^2 \setminus A$ is path-connected.

   (b) Let $B \subseteq \mathbb{R}^2$ be a set of points satisfying the following two conditions:

   (i) if $x \in \mathbb{Q}$, then $(x, y) \in B$ for every $y \in \mathbb{R}$

   (ii) if $x \notin \mathbb{Q}$, then $(x, y) \in B$ for at least one $y \in \mathbb{R}$.

   Show that B is connected.

3. (a) Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $x \in \mathbb{Q}$ if and only if $f(x) \notin \mathbb{Q}$.

   (b) Show that there is no continuous injective function from $\mathbb{R}^2$ to $\mathbb{R}$.

4. Let X be a topological space, and let A be a subset of X. Prove that at most seven distinct sets (including A itself) can be obtained from A by repeated applications of the closure and interior operations.

   Find a subset of $\mathbb{R}$ from which seven distinct sets can be obtained using the closure and interior operations.

5. Let X be $\mathbb{R}$ with the *half-open interval topology*, which has a base of open sets given by all intervals $[a, b)$ with $a < b$. Show that X is totally disconnected, i.e. the only connected subsets are single points. Show also that the interval $[a, b]$, where $a < b$, is closed but not compact.

   Show that $X \times X$ is separable (recall question 10 on sheet 1), but that the subspace $\{(x, -x) : x \in \mathbb{R}\}$ of $X \times X$ is not separable. Deduce that X is not metrizable.

6. Let $\tau_c \subsetneq \tau \subsetneq \tau_f$ be topologies on a set X. (I.e., 'c' for 'coarser' and 'f' for 'finer'.)

   (a) Suppose that (X, τ) is compact. Show that (X, τ_c) is compact. Give an example to show that (X, τ_f) may not be compact.

   (b) Suppose that (X, τ) is Hausdorff. Show that (X, τ_f) is Hausdorff. Give an example to show that (X, τ_c) may not be Hausdorff.

   (c) Suppose that (X, τ) is compact and Hausdorff. Show that (X, τ_f) is not compact, and that (X, τ_c) is not Hausdorff.

7. We generalise question 6 on sheet 1. Let $f : X \to Y$ be a map between topological spaces, and let Γ_f be its graph.

   (a) Show that if Y is Hausdorff and f is continuous then Γ_f is closed.

   (b) Show that if Y is compact then the projection $\pi_1 : X \times Y \to X$, $(x, y) \mapsto x$, is a closed map, i.e. sends closed sets to closed sets.

   Deduce that if Y is compact and Γ_f is closed then f is continuous.

8. A topological space X is called *normal* if, given disjoint closed subsets A, B of X, there exist disjoint open subsets U, V of X with $A \subseteq U$ and $B \subseteq V$. (So question 7 on sheet 1 shows that any metric space is normal.)

   (a) Show that we may choose the open subsets U, V above to have disjoint closures.

   (b) Prove that a compact Hausdorff space is normal.

9. Let U, V be subsets of a topological space X. If U, V are compact, must $U \cup V$ be compact? Must $U \cap V$ be compact?

10. Let X be a finite topological space. Show that if X is Hausdorff then the topology is discrete. Show that if X is connected then X is path-connected.

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11. Let $C_1, C_2, \ldots$ be compact, connected, non-empty subsets of a Hausdorff space, such that $C_1 \supseteq C_2 \supseteq \cdots$. Prove that the intersection $\bigcap_{n \in \mathbb{N}} C_n$ is connected.

    Give an example to show that the compactness assumption is required.

12. Show that the torus $\mathbb{T}^2$ is homeomorphic to $S^1 \times S^1$ with the product topology.

    Let $C = S^1 \times \{1\}$ be a circle in $X = S^1 \times S^1$. Show that X/C is homeomorphic to S^2/A, where $A = \{(0,0,1), (0,0,-1)\}$.

13. Let X, Y be topological spaces such that there exist continuous bijections $f : X \to Y$ and $g : Y \to X$. Show that if X, Y are finite then they are homeomorphic. What if X, Y are infinite?