1. Determine whether the following subsets of $\mathbb{R}^2$ are open, closed, both or neither.

   (i) $\{(x,y) \mid x<0\} \cup \{(x,y) \mid x>0, y>1/x\}$

   (ii) $\{(x,\sin(1/x)) \mid x>0\} \cup \{(0,y) \mid y \in [-1,1]\}$

   (iii) $\{(x,y) \mid y = x^n$ for some positive integer $n\}$.

2. (a) Let (X_1,d_1) and (X_2,d_2) be metric spaces. Show that we may define a metric d on the product $X_1 \times X_2$ by $d((x_1,x_2),(y_1,y_2)) = d_1(x_1,y_1) + d_2(x_2,y_2)$.

   Show that the projections $\pi_i : X_1 \times X_2 \to X_i$, $(x_1,x_2) \mapsto x_i$, are continuous.

   Show that if (X_1,d_2) and (X_2,d_2) are complete, then so is $(X_1 \times X_2, d)$.

   (b) Let (X_i,d_i) be metric spaces for $i=1,2,\ldots$, and let X be the set of all sequences $(x_i)_{i=1}^{\infty}$ with $x_i \in X_i$ for all i. Show that we may define a metric d on X by

$$d\big((x_i),(y_i)\big) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} \; .$$

3. (a) Let d_1, d_2, d_∞ be the metrics on $\mathbb{R}^n$ given by

$$d_1(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|, \quad d_2(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^{n} (x_i-y_i)^2\right]^{1/2}, \quad d_\infty(\mathbf{x},\mathbf{y}) = \sup_i |x_i - y_i|.$$

   For $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, show that

$$d_1(\mathbf{x},\mathbf{y}) \geqslant d_2(\mathbf{x},\mathbf{y}) \geqslant d_\infty(\mathbf{x},\mathbf{y}) \geqslant \frac{1}{\sqrt{n}} d_2(\mathbf{x},\mathbf{y}) \geqslant \frac{1}{n} d_1(\mathbf{x},\mathbf{y}) \; .$$

   Deduce that the metrics induce the same topology on $\mathbb{R}^n$.

   (b) Now let d_1, d_2, d_∞ be the metrics on $C[0,1]$ given by

$$d_1(f,g) = \int_0^1 |f-g|, \quad d_2(f,g) = \left[\int_0^1 (f-g)^2\right]^{1/2}, \quad d_\infty(f,g) = \sup_{[0,1]} |f-g|.$$

   Show that the metrics induce distinct topologies on $C[0,1]$.

4. Define the maps $f,g : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x+y$ and $g(x,y) = xy$. Show that f,g are continuous with respect to the Euclidean topologies on $\mathbb{R}^2$ and $\mathbb{R}$.

   Now give $\mathbb{R}$ the topology τ in which open sets are intervals of the form (a,∞), and $\mathbb{R}^2$ the resulting product topology. Are f,g continuous with respect to these topologies? Can the topology τ be induced by a metric?

   Find all continuous functions from $(\mathbb{R},\tau)$ to $(\mathbb{R}, \text{Euclidean})$.

5. Show that $\mathbb{Q}$ is not complete with respect to the Euclidean metric.

   Is there a metric on $\mathbb{Q}$ which makes it into a complete metric space?

6. For a function $f : X \to Y$, define its *graph* to be $\Gamma_f = \{(x,f(x)) : x \in X\} \subseteq X \times Y$.

   Prove that $f \colon [0,1] \to [0,1]$ is continuous if and only if Γ_f is closed in $[0,1]^2$.

   Give an example of $f : \mathbb{R} \to \mathbb{R}$ for which Γ_f is closed in $\mathbb{R}^2$ but f is not continuous.

7. Let (X, d) be a metric space. For a subset $A \subseteq X$, define $d_A : X \to \mathbb{R}$ by $d_A(x) = \inf_{y \in A} d(x, y)$. Show that d_A is continuous, and that A is closed if and only if $d_A(x) > 0$ for all $x \notin A$.

   Let A, B be disjoint closed subsets of X. Show that there exist disjoint open subsets U, V of X with $A \subseteq U$ and $B \subseteq V$. Must we have $\inf_{x \in B} d_A(x) > 0$?

8. Let $f : X \to Y$ be a map of topological spaces. Show that f is continuous if and only if $f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$ for all $A \subseteq X$.

   Deduce that if f is continuous and surjective, then the image of a dense set in X is dense in Y.

9. Let X be a topological space. Show that the following statements are equivalent:

   (i) X is Hausdorff

   (ii) The diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$, with the product topology

   (iii) For any topological space Y and any continuous maps $f, g : Y \to X$, the set $\{y \in Y : f(y) = g(y)\}$ is closed in Y.

   Deduce that if X is Hausdorff and $f : Y \to X$ is a continuous function on a space Y, then f is determined by its values on any dense subset of Y.

   If instead the diagonal Δ is an open subset of $X \times X$, what is the topology on X?

10. A topological space is called *separable* if it has a countable dense subset, and is called *second countable* if it has a countable base of open sets.

    (a) Show that $\mathbb{R}$ with the Euclidean topology is separable and second countable.

    (b) Let X be $\mathbb{R}$ with the topology in which a subset of $\mathbb{R}$ is open if either it is empty or contains 0. Is X separable? Is X second countable?

    (c) Prove that a second countable topological space is separable, and that a separable metric space is second countable. Deduce that a subspace of a separable metric space is separable.

    (d) Prove also that a product of two separable topological spaces is separable.

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11. Refer back to question 2(b). Prove that if each (X_i, d_i) is complete then so is (X, d), and that if each (X_i, d_i) is separable then so is (X, d).

12. Let X be $\mathbb{R}$ with the *cocountable topology*, in which a subset of $\mathbb{R}$ is open if either it is empty or its complement in $\mathbb{R}$ is countable. Is X separable? Is X second countable? Which sequences $(x_i)_{i=1}^{\infty}$ in X converge, and what is their limit?

    Repeat with the *cofinite topology*, in which a subset of $\mathbb{R}$ is open if either it is empty or its complement in $\mathbb{R}$ is finite.

13. (a) Find a sequence $(a_1, b_1), (a_2, b_2), \ldots$ of open intervals in $\mathbb{R}$ whose union contains all rationals in $[0, 1]$ and such that $\sum_{n=1}^{\infty}(b_n - a_n) < 1$.

    (b) Let $[a_1, b_1], [a_2, b_2], \ldots$ be a sequence of closed intervals in $\mathbb{R}$ whose union contains all of $[0, 1]$. Does it follow that $\sum_{n=1}^{\infty}(b_n - a_n) \geqslant 1$?

1. Let $\mathbb{R}$ have the Euclidean topology, and let $\sim$ be the equivalence relation defined by $x \sim y$ if and only if either $x = y = 0$ or $xy > 0$. Find all open sets of the quotient space $\mathbb{R}/\sim$, and hence determine whether $\mathbb{R}/\sim$ is Hausdorff.

   Now let $\sim$ instead be the equivalence relation on $\mathbb{R}$ defined by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. What is the topology on $\mathbb{R}/\sim$?

2. Determine whether the following subsets of $\mathbb{R}^2$ are connected, path-connected, both or neither. Here, $B_r(\mathbf{x})$ is the open ball with centre $\mathbf{x}$ and radius r, and $\overline{B}_r(\mathbf{x})$ is the corresponding closed ball.

   (i) $B_1((1,0)) \cup B_1((-1,0))$

   (ii) $B_1((1,0)) \cup \overline{B}_1((-1,0))$

   (iii) $\{(x,y) \mid \text{at least one of } x \text{ and } y \text{ is rational}\}$

   (iv) $\{(x,y) \mid \text{exactly one of } x \text{ and } y \text{ is rational}\}$.

3. Let X be a connected topological space.

   (a) Let $f : X \to Y$ be a locally constant map to a topological space Y, i.e. for every $x \in X$, there is an open neighbourhood of x on which f is constant. Show that f is constant.

   (b) Suppose that for every $x \in X$, there is an open neighbourhood of x which is path-connected. Show that X is path-connected.

4. Let X be a topological space.

   (a) Let A_i, $i \in I$, be a collection of connected subsets of X such that $A_i \cap A_j \neq \emptyset$ for all $i, j \in I$. Prove that $\bigcup_{i \in I} A_i$ is connected.

   (b) Prove that if A is a connected subset of X, then $\mathrm{cl}(A)$ is also connected. Deduce that any connected component of X is closed.

   (c) Prove that every connected component of the product $X \times X$ is a product of connected components of X.

5. (a) Is there a metric on $\mathbb{Q}$ which makes it into a connected space? What about $\mathbb{R}\backslash\mathbb{Q}$?

   (b) Is there an infinite compact subset of $\mathbb{Q}$, in the Euclidean topology?

6. Aliens have surrounded the Earth with infinitely many flying saucers, and we cannot see out into space in any direction! Being needed in another galaxy, most of their ships teleport away, leaving only finitely many where they were.

   Can they do this in such a way that we still can't see out into space?

7. (a) Show that a subset A of $\mathbb{R}^n$ is compact if and only if every continuous function from A to $\mathbb{R}$ has bounded image.

   (b) Show that any open cover of $\mathbb{R}^n$ has a countable subcover.

8. A collection $\mathcal{C}$ of subsets of a topological space is said to have the *finite intersection property* if every finite subcollection of $\mathcal{C}$ has non-empty intersection.

   Prove that a topological space is compact if and only if, for every collection of closed subsets with the finite intersection property, the whole collection has non-empty intersection.

9. Let X be a topological space. Its *one-point compactification* X^* is defined as follows. As a set, X^* is the union of X with an additional point denoted by ∞. A subset U of X^* is open if either

   (i) $\infty \notin U$ and U is open in X, or

   (ii) $\infty \in U$ and $X^* \setminus U$ is a closed and compact subset of X.

   Show that this defines a topology, and that X^* is compact. When is X dense in X^*?

   Write down a subset of $\mathbb{R}$ which is homeomorphic to $\mathbb{Z}^*$.

   Show that $\mathbb{C}^*$ is homeomorphic to the sphere S^2.

10. Let $\mathbb{T}^2$ be the two-dimensional torus, defined as $\mathbb{R}^2/\sim$, where $(x, y) \sim (x', y')$ if and only if $x - x'$ and $y - y'$ are both integers.

    (a) Show that $\mathbb{T}^2$ is compact and path-connected.

    (b) Let $L \subset \mathbb{R}^2$ be a line of the form $y = \alpha x$, where α is irrational, and let $q(L)$ be its image in $\mathbb{T}^2$, where $q : \mathbb{R}^2 \to \mathbb{T}^2$ is the quotient map. What are the interior and closure of $q(L)$ in $\mathbb{T}^2$?

    Show that the restriction of q to L is a continuous bijection from L to $q(L)$. Is it a homeomorphism?

    (c) Show that $\mathbb{T}^2$ is homeomorphic to $S^1 \times S^1$ with the product topology.

    Let $C = S^1 \times \{1\}$ be a circle in $X = S^1 \times S^1$. Show that X/C is homeomorphic to S^2/A, where $A = \{(0, 0, 1), (0, 0, -1)\}$.

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11. Refer back to question 2(b) on sheet 1. Prove that if each (X_i, d_i) is connected then so is (X, d), and that if each (X_i, d_i) is compact then so is (X, d).

12. Can a topological space be homeomorphic to its own one-point compactification?

13. For each of the spaces X below, is the one-point compactification X^* metrizable?

    (i) $X = \mathbb{R}$, with the Euclidean topology

    (ii) $X = \mathbb{R}$, with the discrete topology

    (iii) $X = \mathbb{Q}$, with the Euclidean topology

    (iv) $X = \mathbb{Q}$, with the discrete topology.

1. Show that the sequence 2018, 20018, 200018, ... converges in the 5-adic metric on $\mathbb{Z}$.

   Show that the sequence 2018, 22018, 222018, ... is Cauchy but doesn't converge.

2. (a) Let $A \subseteq \mathbb{R}^2$ be a countable set of points. Show that $\mathbb{R}^2 \setminus A$ is path-connected.

   (b) Let $B \subseteq \mathbb{R}^2$ be a set of points satisfying the following two conditions:

   (i) if $x \in \mathbb{Q}$, then $(x, y) \in B$ for every $y \in \mathbb{R}$
   (ii) if $x \notin \mathbb{Q}$, then $(x, y) \in B$ for at least one $y \in \mathbb{R}$.

   Show that B is connected.

3. (a) Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $x \in \mathbb{Q}$ if and only if $f(x) \notin \mathbb{Q}$.

   (b) Show that there is no continuous injective function from $\mathbb{R}^2$ to $\mathbb{R}$.

4. Let X be a topological space, and let A be a subset of X. Prove that at most seven distinct sets (including A itself) can be obtained from A by repeated applications of the closure and interior operations.

   How many distinct sets can be obtained if we also allow the complement operation?

   Find a subset of $\mathbb{R}$ from which seven distinct sets can be obtained using the closure and interior operations.

5. Let X be $\mathbb{R}$ with the *half-open interval topology*, which has a base of open sets given by all intervals $[a, b)$ with $a < b$. Show that X is totally disconnected, i.e. the only connected subsets are single points. Show also that the interval $[a, b]$, where $a < b$, is closed but not compact.

   Show that $X \times X$ is separable (see question 10 on sheet 1), but that the subspace $\{(x, -x) : x \in \mathbb{R}\}$ of $X \times X$ is not separable. Deduce that X is not metrizable.

6. A topological space X is called *normal* if, given disjoint closed subsets A, B of X, there exist disjoint open subsets U, V of X with $A \subseteq U$ and $B \subseteq V$. (So question 7 on sheet 1 shows that any metric space is normal.)

   (a) Show that we may choose the open subsets U, V above to have disjoint closures.

   (b) Show that if X is normal and $C \subseteq X$ is closed, then the quotient X/C is normal.

   (c) Prove that a compact Hausdorff space is normal.

7. We generalise question 6 on sheet 1. Let $f : X \to Y$ be a map between topological spaces, and let Γ_f be its graph.

   (a) Show that if Y is Hausdorff and f is continuous then Γ_f is closed.

   (b) Show that if Y is compact then the projection $\pi_1 : X \times Y \to X$, $(x, y) \mapsto x$, is a closed map, i.e. sends closed sets to closed sets.

   Deduce that if Y is compact and Γ_f is closed then f is continuous.

8. Let $\tau_c \subsetneq \tau \subsetneq \tau_f$ be topologies on a set X. (I.e., 'c' for 'coarser' and 'f' for 'finer'.)

   (a) Suppose that (X, τ) is compact. Show that (X, τ_c) is compact. Give an example to show that (X, τ_f) may not be compact.

   (b) Suppose that (X, τ) is Hausdorff. Show that (X, τ_f) is Hausdorff. Give an example to show that (X, τ_c) may not be Hausdorff.

   (c) Suppose that (X, τ) is compact and Hausdorff. Show that (X, τ_f) is not compact, and that (X, τ_c) is not Hausdorff.

9. Let U, V be subsets of a topological space X. If U, V are compact, must $U \cup V$ be compact? Must $U \cap V$ be compact?

10. Let X be a finite topological space. Show that if X is Hausdorff then the topology is discrete. Show that if X is connected then X is path-connected.

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11. Let $C_1, C_2, \ldots$ be compact, connected, non-empty subsets of a Hausdorff space, such that $C_1 \supseteq C_2 \supseteq \cdots$. Prove that the intersection $\bigcap_{n \in \mathbb{N}} C_n$ is connected.

    Give examples to show that the compact and Hausdorff assumptions are both required.

12. Let X, Y be topological spaces such that there exist continuous bijections $f : X \to Y$ and $g : Y \to X$. Must X and Y be homeomorphic?

*13. Is there a continuous bijection from $\mathbb{R}$ to $\mathbb{R}^2$?