A short thing about determinants

Or "an attempt to explain volume forms, determinants and adjugates by staying in 3-d"

We are going to try to define the volume of a parallelepiped spanned by three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. By 'spanned', I mean that the parallelepiped has its corners at

$$\mathbf{0},\ \mathbf{a},\ \mathbf{b},\ \mathbf{c},\ \mathbf{a}+\mathbf{b},\ \mathbf{a}+\mathbf{c},\ \mathbf{b}+\mathbf{c},\ \mathbf{a}+\mathbf{b}+\mathbf{c}$$

We'll define a function D such that $D(\mathbf{a}, \mathbf{b}, \mathbf{c})$ gives us the volume of the parallelepiped in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. We want to do this in such a way that it extends to higher dimensions, so let's not think about any volume formulae we know (like 'base area times height'). Instead, let's think about what sort of properties any function that tries to be 'volume' ought to satisfy. For example:

(i) If any two of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are equal, then $D(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$. This makes sense: the cuboid is then just a flat parallelogram, of volume 0.

(ii) If we have two parallelepipeds with common base area (say, spanned by $\mathbf{b}$ and $\mathbf{c}$), then we could imagine gluing them together on their shared base, and shearing the parts to make their other faces parallel. Since shears don't change volume, the resulting parallepiped's volume is then the sum of the two initial volumes. That is, D should satisfy $D(\mathbf{a}+\mathbf{a}', \mathbf{b}, \mathbf{c}) = D(\mathbf{a}, \mathbf{b}, \mathbf{c}) + D(\mathbf{a}', \mathbf{b}, \mathbf{c})$, and similarly in the other components.

(iii) Similarly, if we scale one dimension of the parallelepiped then we should scale the volume by the same amount. That is, D should satisfy $D(k\mathbf{a}, \mathbf{b}, \mathbf{c}) = kD(\mathbf{a}, \mathbf{b}, \mathbf{c})$, and similarly in the other components.

Note that (ii) and (iii) say that D should be a *linear* function in each component.

An immediate consequence of these is that $D(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$ if any two of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are parallel rather than actually equal. E.g., consider $D(\lambda\mathbf{a}, \mu\mathbf{a}, \mathbf{c})$. By (ii) we can pull out the factors of λ and μ. This then equals $\lambda\mu D(\mathbf{a}, \mathbf{a}, \mathbf{c})$, which is 0 by (i).

We also have a very useful result about permuting the vectors we feed to D.

By rule (i), $D(\mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b}, \mathbf{c}) = 0$. $(*)$

By rule (ii) in the 1st argument: $(*) = D(\mathbf{a}, \mathbf{a}+\mathbf{b}, \mathbf{c}) + D(\mathbf{b}, \mathbf{a}+\mathbf{b}, \mathbf{c})$.

By rule (ii) in both 2nd arguments: $(*) = D(\mathbf{a}, \mathbf{a}, \mathbf{c}) + D(\mathbf{a}, \mathbf{b}, \mathbf{c}) + D(\mathbf{b}, \mathbf{a}, \mathbf{c}) + D(\mathbf{b}, \mathbf{b}, \mathbf{c})$.

By rule (i), the 1st and 4th terms here = 0. So, we are left with $D(\mathbf{a}, \mathbf{b}, \mathbf{c}) + D(\mathbf{b}, \mathbf{a}, \mathbf{c}) = 0$. I.e.

$$D(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -D(\mathbf{b}, \mathbf{a}, \mathbf{c}).$$

Similarly if we'd used the first and third arguments, or the second and third. So if we swap any two arguments, the value of D negates. As a consequence, if we cycle the three arguments, the sign doesn't change, since a cycle of three things can be performed with two swaps, so we get two negations.

$$\begin{aligned} \text{E.g.,}\quad D(\mathbf{b}, \mathbf{c}, \mathbf{a}) &= -D(\mathbf{b}, \mathbf{a}, \mathbf{c}) \quad \text{(swap 2nd \& 3rd arguments)} \\ &= D(\mathbf{a}, \mathbf{b}, \mathbf{c}) \quad \text{(swap 1st \& 2nd arguments)} \end{aligned}$$

Now suppose that we have written $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in terms of a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, i.e.

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3,\ \ \mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3,\ \ \mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$$

We then apply the summation rule in (ii) many times to expand this all out:

$$\begin{aligned} D(\mathbf{a},\mathbf{b},\mathbf{c}) &= D(a_1\mathbf{e}_1 + a_2\mathbf{e}_3 + a_3\mathbf{e}_3, b_1\mathbf{e}_1 + b_2\mathbf{e}_3 + b_3\mathbf{e}_3, c_1\mathbf{e}_1 + c_2\mathbf{e}_3 + c_3\mathbf{e}_3) \\ &= D(a_1\mathbf{e}_1, b_1\mathbf{e}_1, c_1\mathbf{e}_1) + D(a_1\mathbf{e}_1, b_1\mathbf{e}_1, c_2\mathbf{e}_2) + \cdots + D(a_3\mathbf{e}_3, b_3\mathbf{e}_3, c_3\mathbf{e}_3) \end{aligned}$$

There are 27 terms here, because we expand three terms in three components. However, most are 0, because they have parallel arguments. In fact, only six terms are left:

$$\begin{aligned} D(\mathbf{a},\mathbf{b},\mathbf{c}) &= D(a_1\mathbf{e}_1, b_2\mathbf{e}_2, c_3\mathbf{e}_3) + D(a_1\mathbf{e}_1, b_3\mathbf{e}_3, c_2\mathbf{e}_2) + D(a_2\mathbf{e}_2, b_3\mathbf{e}_3, c_1\mathbf{e}_1) \\ &\quad + D(a_2\mathbf{e}_2, b_1\mathbf{e}_1, c_3\mathbf{e}_3) + D(a_3\mathbf{e}_3, b_1\mathbf{e}_1, c_2\mathbf{e}_2) + D(a_3\mathbf{e}_3, b_2\mathbf{e}_2, c_1\mathbf{e}_1) \end{aligned}$$

We now use the second part of (ii) to pull these coefficients out:

$$\begin{aligned} D(\mathbf{a},\mathbf{b},\mathbf{c}) &= a_1b_2c_3\, D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) + a_1b_3c_2\, D(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_2) + a_2b_3c_1\, D(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1) \\ &\quad + a_2b_1c_3\, D(\mathbf{e}_2,\mathbf{e}_1,\mathbf{e}_3) + a_3b_1c_2\, D(\mathbf{e}_3,\mathbf{e}_1,\mathbf{e}_2) + a_3b_2c_1\, D(\mathbf{e}_3,\mathbf{e}_2,\mathbf{e}_1) \end{aligned}$$

We'll now use our result about swapping and cycling the terms in the D. E.g., the second term has $D(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_2)$, which needs a single swap, so equals $-D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$. We'll permute term until each is ordered as $\pm D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$. We get

$$\begin{aligned} D(\mathbf{a},\mathbf{b},\mathbf{c}) &= \big(a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1\big) D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) \end{aligned}$$

(Note that if we were to impose the additional condition that $D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) = 1$, then we see that the only choice for $D(\mathbf{a},\mathbf{b},\mathbf{c})$ is as the determinant of the matrix with rows $\mathbf{a},\mathbf{b},\mathbf{c}$.)

We could use the reasoning above to motivate the definition of the determinant in the first place. That is, we choose to define

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

since it gives the 'volume scale factor' as we saw above.

If you've met suffix notation, note that this can be written as $\epsilon_{ijk}a_ib_jc_k$. (Check this.)

Suppose we now let

$$\mathbf{p} = p_1\mathbf{a} + p_2\mathbf{b} + p_3\mathbf{c},\ \ \mathbf{q} = q_1\mathbf{a} + q_2\mathbf{b} + q_3\mathbf{c},\ \ \mathbf{r} = r_1\mathbf{a} + r_2\mathbf{b} + r_3\mathbf{c}$$

Then every step in the above working still holds in terms of these, and we find

$$\begin{aligned} D(\mathbf{p},\mathbf{q},\mathbf{r}) &= \big(p_1q_2r_3 - p_1q_3r_2 + p_2q_3r_1 - p_2q_1r_3 + p_3q_1r_2 - p_3q_2r_1\big) D(\mathbf{a},\mathbf{b},\mathbf{c}) \\ &= \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} D(\mathbf{a},\mathbf{b},\mathbf{c}) \;=\; \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) \end{aligned}$$

But note also that, with an abuse of notation for shorthand purposes,

$$\begin{aligned} \begin{pmatrix} \leftarrow & \mathbf{p} & \rightarrow \\ \leftarrow & \mathbf{q} & \rightarrow \\ \leftarrow & \mathbf{r} & \rightarrow \end{pmatrix} &= \begin{pmatrix} \leftarrow & p_1\mathbf{a} + p_2\mathbf{b} + p_3\mathbf{c} & \rightarrow \\ \leftarrow & q_1\mathbf{a} + q_2\mathbf{b} + q_3\mathbf{c} & \rightarrow \\ \leftarrow & r_1\mathbf{a} + r_2\mathbf{b} + r_3\mathbf{c} & \rightarrow \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix} \begin{pmatrix} \leftarrow & \mathbf{a} & \rightarrow \\ \leftarrow & \mathbf{b} & \rightarrow \\ \leftarrow & \mathbf{c} & \rightarrow \end{pmatrix} \\ &= \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} \leftarrow & \mathbf{e}_1 & \rightarrow \\ \leftarrow & \mathbf{e}_2 & \rightarrow \\ \leftarrow & \mathbf{e}_3 & \rightarrow \end{pmatrix} \end{aligned}$$

That is,

$$\mathbf{p} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3,\ \ \mathbf{q} = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2 + \beta_3\mathbf{e}_2,\ \ \mathbf{r} = \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2 + \gamma_3\mathbf{e}_3$$

where

$$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{pmatrix} = \begin{pmatrix}p_1 & p_2 & p_3\\ q_1 & q_2 & q_3\\ r_1 & r_2 & r_3\end{pmatrix}\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}$$

Applying the earlier result again, we find

$$D(\mathbf{p},\mathbf{q},\mathbf{r}) \ = \begin{vmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{vmatrix} D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$$

and comparing with the earlier expression for $D(\mathbf{p},\mathbf{q},\mathbf{r})$ we see that

$$\begin{vmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{vmatrix} = \begin{vmatrix}p_1 & p_2 & p_3\\ q_1 & q_2 & q_3\\ r_1 & r_2 & r_3\end{vmatrix}\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

(assuming that $D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) \neq 0$, but we may assume, or even define, this to be true).

In other words, we have shown that

$$\det(AB) = \det A \det B$$

The Adjugate Matrix

The adjugate matrix of A is obtained by taking the cofactors of A, then putting these into a matrix and transposing it. That is:

$$\text{If}\ \ A = \begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}\ \ \text{then}\ \ \text{adj}A = \begin{pmatrix} \begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix} & -\begin{vmatrix}a_2 & a_3\\ c_2 & c_3\end{vmatrix} & \begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\\ -\begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix} & \begin{vmatrix}a_1 & a_3\\ c_1 & c_3\end{vmatrix} & -\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\\ \begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix} & -\begin{vmatrix}a_1 & a_2\\ c_1 & c_2\end{vmatrix} & \begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\end{pmatrix}$$

Let's compute some entries of $A\,\text{adj}A$.

$$\begin{aligned}(A\,\text{adj}A)_{11} &= a_1\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix} - a_2\begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix} + a_3\begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}\\ \textit{(expand along 1}^{st}\textit{ row)} &= \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} = \det A\end{aligned}$$

$$\begin{aligned}(A\,\text{adj}A)_{21} &= b_1\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix} - b_2\begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix} + b_3\begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}\\ \textit{(expand along 1}^{st}\textit{ row)} &= \begin{vmatrix}b_1 & b_2 & b_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} = 0\end{aligned}$$

$$
\begin{aligned}
(A\,\mathrm{adj}A)_{33} &= c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\
\textit{(expand along } 3^{rd} \textit{ row)} &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \det A
\end{aligned}
$$

And so on. In general, the diagonal entries equal $\det A$ and the rest are 0. In other words, $A\,\mathrm{adj}A = I\,\det A$.

Higher dimensions

If we were to repeat our method from the first section with a four-dimensional 'parallelepiped' spanned by **a**, **b**, **c**, **d**, then the reasoning would go through in exactly the same way. We would find that

$$D(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = (a_1b_2c_3d_4 - a_1b_2c_4d_3 - a_1b_3c_2d_4 + a_1b_3c_4d_2 - \ldots)D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4)$$

There would be 24 terms in this list, namely one for each permutation of the subscripts. We can't use the same suffix notation formula we used for three dimensions earlier, but we can extend it.

As before, the sign on each term in the sum is given by the number of swaps required to return the permutation to the order 1234.

Let $\sigma = (34)$ be a transposition. Its sign is $\epsilon(\sigma) = -1$. So we may write the second term in the sum above as

$$-a_1b_2c_4d_3 = \epsilon(\sigma)a_{\sigma(1)}b_{\sigma(2)}c_{\sigma(3)}d_{\sigma(4)}$$

Similarly, with $\sigma = (234)$, the fourth term in the sum is

$$+a_1b_3c_4d_2 = \epsilon(\sigma)a_{\sigma(1)}b_{\sigma(2)}c_{\sigma(3)}d_{\sigma(4)}$$

Indeed, every term in the sum may be written as $\epsilon(\sigma)a_{\sigma(1)}b_{\sigma(2)}c_{\sigma(3)}d_{\sigma(4)}$ for a suitable permutation σ. And since every permutation of 1234 occurs, we may therefore write the whole as

$$\sum_{\sigma\in S_4} \epsilon(\sigma)a_{\sigma(1)}b_{\sigma(2)}c_{\sigma(3)}d_{\sigma(4)}$$

As before, we will choose to define this to be the determinant of the 4×4 matrix with rows given by **a**, **b**, **c**, **d**.

This procedure can, of course, continue into even higher dimensions. But eventually it becomes silly to call the rows by different letters. Instead, rather than $\mathbf{a} = (a_1, \ldots, a_n)$, the top row would be $(a_{11}, \ldots, a_{1n})$, and the i^{th} row would be $(a_{i1}, \ldots, a_{in})$.

The above formula for the determinant would then be:

$$\sum_{\sigma\in S_n} \epsilon(\sigma)a_{1\sigma(1)}\ldots a_{i\sigma(i)}\ldots a_{n\sigma(n)}$$

The results $\det(AB) = \det A \det B$ and $A\,\mathrm{adj}(A) = I \det A$ still hold, and the proofs are much the same (only with general subscripts and sums, rather than explicitly with three dimensions).

A useful lemma

Suppose that $A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & c_3 & c_4 \\ 0 & 0 & d_3 & d_4 \end{pmatrix}$.

If we expand this determinant down its first column, we obtain

$$\begin{aligned} \det A &= a_1 \begin{vmatrix} b_2 & b_3 & c_3 \\ 0 & c_3 & c_4 \\ 0 & d_3 & d_4 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 & a_3 \\ 0 & c_3 & c_4 \\ 0 & d_3 & d_4 \end{vmatrix} \\ &= a_1 b_2 \begin{vmatrix} c_3 & c_4 \\ d_3 & d_4 \end{vmatrix} - b_1 a_2 \begin{vmatrix} c_3 & c_4 \\ d_3 & d_4 \end{vmatrix} \\ &= (a_1 b_2 - b_1 a_2) \begin{vmatrix} c_3 & c_4 \\ d_3 & d_4 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} c_3 & c_4 \\ d_3 & d_4 \end{vmatrix} \end{aligned}$$

More generally, if $A = \begin{pmatrix} P & Q \\ O & R \end{pmatrix}$ with P, Q square matrices and O a zero matrix, we have $\det A = \det P \det R$.

We'll show why this is true in the above 4×4 case. As usual, the proof extends to the general case, just with more terms. So suppose A is as at the top of the page.

In the usual sum for the determinant of a 4×4 matrix, there are 24 terms – one for each permutation of the subscripts. But here, if we ever have $\sigma(3)$ or $\sigma(4)$ equalling 1 or 2, then that summand is 0, as it contains one of c_1, c_2, d_1, d_2.. So the only $\sigma \in S_4$ which contribute to the sum are which σ sends 3 and 4 to 3 and 4 (in some order), and hence sends 1 and 2 to 1 and 2 (in some order).

Such a σ equals $\tau\tau'$, where τ is a permutation of $\{1, 2\}$, and τ' is a permutation of $\{3, 4\}$.

We obtain, using halfway through that $\epsilon(\tau\tau') = \epsilon(\tau)\epsilon(\tau')$, the following:

$$\begin{aligned} \det A &= \sum_{\sigma \in S_4} \epsilon(\sigma) a_{\sigma(1)} b_{\sigma(2)} c_{\sigma(3)} d_{\sigma(4)} \\ &= \sum_{\substack{\tau \in S_{\{1,2\}} \\ \tau' \in S_{\{3,4\}}}} \epsilon(\tau\tau') a_{\sigma(1)} b_{\sigma(2)} c_{\sigma(3)} d_{\sigma(4)} \\ &= \sum_{\tau \in S_{\{1,2\}}} \epsilon(\tau) a_{\sigma(1)} b_{\sigma(2)} \sum_{\tau \in S_{\{3,4\}}} \epsilon(\tau') c_{\sigma(3)} d_{\sigma(4)} \\ &= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} c_3 & c_4 \\ d_3 & d_4 \end{vmatrix} \end{aligned}$$

since the second sum in the third row was the usual 2×2 matrix formula just in terms of 3 and 4, rather than 1 and 2.

Warning! If $A = \begin{pmatrix} P & Q \\ S & R \end{pmatrix}$ with S *not* a zero matrix, then we *do not* have $\det A = \det P \det R - \det Q \det S$. Don't think that the usual 2×2 matrix formula can be applied to block matrices!

Please let me know of any corrections: glt1000@cam.ac.uk