

IB Linear Algebra – Example Sheet 2

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1. Use elementary row operations to prove that the real matrix:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}$$

is invertible, and find its inverse.

2. Let V be an n -dimensional vector space over the field $\mathbb{F}$; let B be a basis for V , and let $P \in \text{GL}_n(\mathbb{F})$ be an invertible n -by- n matrix over $\mathbb{F}$. Prove that there are bases C and D for V such that the change-of-basis matrices $[id_V]_B^C$ and $[id_V]_D^B$ are equal to P .
3. (Another proof of the row rank column rank equality.) Let A be an $m \times n$ matrix of (column) rank r . Show that r is the least integer for which A factorises as $A = BC$ with $B \in \mathbb{M}_{m \times r}(\mathbb{F})$ and $C \in \mathbb{M}_{r \times n}(\mathbb{F})$.

Using the fact that $(BC)^T = C^T B^T$, deduce that the (column) rank of A^T equals r .

4. Let A and B be $n \times n$ matrices over a field $\mathbb{F}$. Show that the $2n \times 2n$ matrix

$$C = \begin{pmatrix} I & -B \\ A & 0 \end{pmatrix} \text{ can be transformed into } D = \begin{pmatrix} I & 0 \\ A & AB \end{pmatrix}$$

by elementary column operations (which you should specify). By considering the determinants of C and D , obtain another proof that $\det(AB) = \det(A)\det(B)$.

5. Let $\alpha : V \rightarrow V$ be a linear map on a real finite dimensional vector space V with $\text{tr}(\alpha) = 0$.
 - (i) Show that, if $\alpha \neq 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
 - (ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.
6. Let A, B be $n \times n$ matrices over the field $\mathbb{F}$, where $n \geq 2$. Show that, if A and B are non-singular, then:

$$(i) \text{ adj}(AB) = \text{adj}(B)\text{adj}(A), \quad (ii) \det(\text{adj } A) = (\det A)^{n-1}, \quad (iii) \text{ adj}(\text{adj } A) = (\det A)^{n-2}A.$$

Show that the rank of the adjugate matrix is $\text{rk}(\text{adj } A) = \begin{cases} n & \text{if } \text{rk}(A) = n \\ 1 & \text{if } \text{rk}(A) = n - 1 \\ 0 & \text{if } \text{rk}(A) \leq n - 2. \end{cases}$

Do (i)-(iii) hold if A is singular? [Hint: for (i) consider $A + \lambda I$ for $\lambda \in \mathbb{F}$.]

7. Let V be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of V^* dual to the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ for V . Determine, in terms of the ξ_i , the bases dual to each of the following:
 - (a) $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3\}$;
 - (b) $\{\mathbf{x}_1, 2\mathbf{x}_2, \frac{1}{2}\mathbf{x}_3, \mathbf{x}_4\}$;
 - (c) $\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4\}$;
 - (d) $\{\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3 + \mathbf{x}_2 - \mathbf{x}_1\}$.
8. For $A \in \mathbb{M}_{n \times m}(\mathbb{F})$ and $B \in \mathbb{M}_{m \times n}(\mathbb{F})$, let $\tau_A(B)$ denote $\text{tr}(AB)$. Show that, for each fixed A , $\tau_A : \mathbb{M}_{m \times n}(\mathbb{F}) \rightarrow \mathbb{F}$ is linear. Show moreover that the mapping $A \mapsto \tau_A$ defines a linear isomorphism $\mathbb{M}_{n \times m}(\mathbb{F}) \rightarrow \mathbb{M}_{m \times n}(\mathbb{F})^*$.

9. (a) Suppose that $f \in \mathbb{M}_{n \times n}(\mathbb{R})^*$ is such that $f(AB) = f(BA)$ for all $A, B \in \mathbb{M}_{n \times n}(\mathbb{R})$ and $f(I) = n$. Show that f is the trace functional, i.e. $f(A) = \text{tr} A$ for all $A \in \mathbb{M}_{n \times n}(\mathbb{R})$.

(b) Now let V be a non-zero finite dimensional real vector space. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = id_V$.

(c) Let V be the space of infinitely differentiable functions $\mathbb{R} \rightarrow \mathbb{R}$. Find endomorphisms α and β of V such that $\alpha\beta - \beta\alpha = id_V$.

10. (a) Let $a_0, \dots, a_n$ be distinct real numbers, and let

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{pmatrix}.$$

Show that $\det(A) \neq 0$.

(b) Let P_n be the space of real polynomials of degree at most n . For $x \in \mathbb{R}$ define $e_x \in P_n^*$ by $e_x(p) = p(x)$. By considering the standard basis $(1, t, \dots, t^n)$ for P_n , use (a) to show that $\{e_0, \dots, e_n\}$ is linearly independent and hence forms a basis for P_n^* .

(c) Identify the basis of P_n to which $(e_0, \dots, e_n)$ is dual.

11. Show that the dual of the space P of real polynomials is isomorphic to the space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers, via the mapping which sends a linear form $\xi : P \rightarrow \mathbb{R}$ to the sequence $(\xi(1), \xi(t), \xi(t^2), \dots)$.

In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \dots)$ of the linear maps dual to each of the following linear maps $P \rightarrow P$:

(a) The map D defined by $D(p)(t) = p'(t)$.

(b) The map S defined by $S(p)(t) = p(t^2)$.

(c) The composite DS .

(d) The composite SD .

Verify that $(DS)^* = S^*D^*$ and $(SD)^* = D^*S^*$.

12. Let V be a finite dimensional vector space. Suppose that $f_1, \dots, f_n, g \in V^*$. Show that g is in the span of $f_1, \dots, f_n$ if and only if $\bigcap_{i=1}^n \ker f_i \subset \ker g$. What if V is infinite dimensional?