1. For each case below, let U be the subset of the real vector space $\mathbb{R}^3$ consisting of all vectors (x_1, x_2, x_3) satisfying the given condition. In which cases is U a subspace of $\mathbb{R}^3$?

   (a) $x_1 \geqslant 0$
   (b) either $x_1 = 0$ or $x_2 = 0$
   (c) $x_1 + x_2 = 0$
   (d) $x_1 + x_2 = 1$
   (e) $x_1^2 + x_2^2 = 0$
   (f) $x_1 + x_2 + x_3 = 0$ and $x_1 - x_3 = 0$.

2. Let $\mathbb{R}^{\mathbb{N}}$ be the set of all real sequences, which you may assume is a vector space over $\mathbb{R}$. Determine which of the following sets of sequences of real numbers (x_n) form subspaces of $\mathbb{R}^{\mathbb{N}}$. (You may assume any results from Part IA Analysis I.)

   (a) x_n is bounded
   (b) x_n is convergent
   (c) $x_n \to 0$ as $n \to \infty$
   (d) $x_n \to \infty$ as $n \to \infty$
   (e) $x_{n+2} = x_{n+1} + x_n$ for all n
   (f) there exists N such that $x_n = 0$ for $n > N$
   (g) $\sum |x_n|$ is convergent
   (h) $\sum x_n^2$ is convergent.

3. Let $\mathbb{R}^{\mathbb{R}}$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$, with addition and scalar multiplication defined pointwise, which you may assume is a vector space over $\mathbb{R}$. Determine which of the following sets of functions f form subspaces of $\mathbb{R}^{\mathbb{R}}$.

   (a) f is a polynomial
   (b) f is a polynomial of even degree
   (c) f is constant on $\mathbb{Z}$
   (d) f is a solution of $(f'(t))^2 - f(t) = 0$
   (e) f is a solution of $(f''(t))^4 + (f(t))^2 = 0$
   (f) f is periodic.

4. For each of the vector spaces found in questions 1–3, determine whether it is finite-dimensional or not. When it is finite-dimensional, state the dimension and find a basis. When it is not finite-dimensional, demonstrate why is it not.

5. Which of the following are bases for the given spaces?

   (a) For $\mathbb{R}^3$: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

   (b) For $\mathbb{R}^4$: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

   (c) For $\mathbb{R}^3$: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

   (d) For $\mathbb{R}^4$: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

6. For each following matrix A, find bases for the kernel and image of the linear map $\mathbf{x} \mapsto A\mathbf{x}$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

7. (a) Let P denote the vector space of all polynomial functions $\mathbb{R} \to \mathbb{R}$, and let P_n denote the subspace of P consisting of polynomials of degree at most n. Which of the following define linear maps $P_n \to P$?

   (i) $S(p)(t) = p(t^2 + 1)$
   (ii) $T(p)(t) = p(t)^2 + 1$
   (iii) $U(p)(t) = p(t^2) - tp(t)$
   (iv) $E(p)(t) = p(e^t)$
   (v) $D(p)(t) = p'(t)$
   (vi) $I(p)(t) = \int_0^t p(s)ds$.

   For those that are linear maps, find their rank and nullity.

   (b) Let $Q(p)$ and $R(p)$, respectively, be the quotient and remainder when p is divided by $t^2 + 1$. Show that Q and R are linear maps $P_n \to P$. Find their rank and nullity.

8. The linear map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} -5x + 9y \\ -4x + 7y \end{pmatrix}.$$

   Find the matrix of α relative to the basis $\left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

   Write down the matrix of α relative to the basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$.

9. Let V_1, V_2 be subspaces of $\mathbb{R}^4$ with bases

$$V_1 : \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \\ -2 \\ 3 \end{pmatrix} \right\} \quad \text{and} \quad V_2 : \left\{ \begin{pmatrix} 1 \\ 0 \\ -4 \\ -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \\ 2 \end{pmatrix} \begin{pmatrix} -4 \\ 4 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

   Find a basis for the subspace $V_1 \cap V_2$ of the form $\left\{ \begin{pmatrix} a \\ b \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ d \\ 0 \\ 1 \end{pmatrix} \right\}$, for suitable a, b, c, d.

   (You should find that a, b, c, d are all small integers.)

10. Let A, B be $n \times n$ real matrices. If $AB = 0$, must $BA = 0$? If $(AB)^n = 0$, must $(BA)^n = 0$?

11. (Optional.) A *magic square* is a square matrix whose rows, columns and two main diagonals all sum to the same total. Convince yourself that the set of 3×3 magic squares with real entries form a vector space over $\mathbb{R}$. What is its dimension? Find a nice basis.