



6. Find the ranks of the following matrices  $A$ , and give bases for the kernel and image of the linear maps  $\mathbf{x} \mapsto A\mathbf{x}$ .

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

7. (a) Let  $P$  denote the vector space of all polynomial functions  $\mathbb{R} \rightarrow \mathbb{R}$ , and let  $P_n$  denote the subspace of  $P$  consisting of polynomials of degree at most  $n$ . Which of the following define linear maps  $P_n \rightarrow P$ ?

- (i)  $S(p)(t) = p(t^2 + 1)$                       (iv)  $E(p)(t) = p(e^t)$   
(ii)  $T(p)(t) = p(t)^2 + 1$                       (v)  $D(p)(t) = p'(t)$   
(iii)  $U(p)(t) = p(t^2) - tp(t)$                       (vi)  $I(p)(t) = p(0) + \int_0^t p(s)ds$ .

For those that are linear maps, find their rank and nullity.

- (b) Let  $Q(p)$  and  $R(p)$ , respectively, be the quotient and remainder when  $p$  is divided by  $t^2 + 1$ . Show that  $Q$  and  $R$  are linear maps  $P_n \rightarrow P$ . Find their rank and nullity.

8. The linear map  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -5x + 9y \\ -4x + 7y \end{pmatrix}.$$

Find the matrix of  $\alpha$  relative to the basis

$$\left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Write down the matrix of  $\alpha$  relative to the basis

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}.$$

9. Let  $V_1, V_2$  be subspaces of  $\mathbb{R}^4$  with bases

$$V_1 : \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ -2 \\ 3 \end{pmatrix} \right\} \quad \text{and} \quad V_2 : \left\{ \begin{pmatrix} 1 \\ 0 \\ -4 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 4 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

Find a basis for the subspace  $V_1 \cap V_2$  of the form  $\left\{ \begin{pmatrix} a \\ b \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ d \\ 0 \\ 1 \end{pmatrix} \right\}$ , for suitable  $a, b, c, d$ .

10. Let  $A, B$  be  $n \times n$  real matrices. If  $AB = 0$ , must  $BA = 0$ ? If  $(AB)^n = 0$ , must  $(BA)^n = 0$ ?