Linear Algebra: Example Sheet 2 of 4

1. *(Another proof of the row rank column rank equality.) Let A be an $m \times n$ matrix of (column) rank r. Show that r is the least integer for which A factorises as $A = BC$ with $B \in \mathrm{Mat}_{m,r}(\mathbb{F})$ and $C \in \mathrm{Mat}_{r,n}(\mathbb{F})$. Using the fact that $(BC)^T = C^T B^T$, deduce that the (column) rank of A^T equals r.*

2. *Write down the three types of elementary matrices and find their inverses. Show that an $n \times n$ matrix A is invertible if and only if it can be written as a product of elementary matrices. Write the following matrices as products of elementary matrices. Then use this to find their inverses.*

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}.$$

3. *Let $\lambda \in F$. Evaluate the determinant of the $n \times n$ matrix A with each diagonal entry equal to λ and all other entries 1.*

4. *Let A be an $n \times m$ matrix. Prove that if B is an $m \times n$ matrix then $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.*

5. *Let A and B be $n \times n$ matrices over a field F. Show that the $2n \times 2n$ matrix*

$$C = \begin{pmatrix} I & B \\ -A & 0 \end{pmatrix} \quad \text{can be transformed into} \quad D = \begin{pmatrix} I & B \\ 0 & AB \end{pmatrix}$$

by elementary row operations (which you should specify). By considering the determinants of C and D, obtain another proof that $\det AB = \det A \det B$.

6. *(i) Let V be a non-trivial real vector space of finite dimension. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = \mathrm{id}_V$.*
(ii) Let V be the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Find endomorphisms α, β of V which do satisfy $\alpha\beta - \beta\alpha = \mathrm{id}_V$.

7. *Compute the characteristic polynomials of the matrices*

$$\begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Which of the matrices are diagonalisable over $\mathbb{C}$? Which over $\mathbb{R}$?

8. *Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:*

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

9. *Let $a_0, ..., a_n$ be distinct real numbers, and let*

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{pmatrix}.$$

Show that $\det(A) \neq 0$.

10. *For* $n \geq 2$*, let* $A, B \in \mathrm{Mat}_n(\mathbb{F})$*. Show that, if* A *and* B *are invertible, then*

$$(i)\ \mathrm{adj}\,(AB) = \mathrm{adj}\,(B)\mathrm{adj}\,(A)\,, \quad (ii)\ \det(\mathrm{adj}\,A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}\,(\mathrm{adj}\,A) = (\det A)^{n-2}A\,.$$

What happens if A *is not invertible? Now show that*

$$\mathrm{rank}(\mathrm{adj}\,A) = \begin{cases} n & \textit{if} \quad \mathrm{rank}(A) = n \\ 1 & \textit{if} \quad \mathrm{rank}(A) = n-1 \\ 0 & \textit{if} \quad \mathrm{rank}(A) \leq n-2. \end{cases}$$

11. *Let* V *be a vector space, let* $\pi_1, \pi_2, \ldots, \pi_k$ *be endomorphisms of* V *such that* $\mathrm{id}_V = \pi_1 + \cdots + \pi_k$ *and* $\pi_i\pi_j = 0$ *for any* $i \neq j$*. Show that* $V = U_1 \oplus \cdots \oplus U_k$*, where* $U_j = \mathrm{Im}(\pi_j)$*.*
Let α *be an endomorphism on the vector space* V*, satisfying the equation* $\alpha^3 = \alpha$*. Prove directly that* $V = V_0 \oplus V_1 \oplus V_{-1}$*, where* V_λ *is the* λ*-eigenspace of* α*.*

12. *Let* $A \in \mathrm{Mat}_n(\mathbb{R})$ *be such that for all* $i \in \{1, \ldots, n\}$ *we have* $|a_{i,i}| > \sum_{j \neq i} |a_{i,j}|$*. Show that* A *is invertible. Now additionally assume* $a_{i,i} \geq 0$*, for all* i*; does this imply* $\det(A) > 0$*?*

13. *Let* $A \in \mathrm{Mat}_n(\mathbb{C})$ *satisfy* $A^k = I$*, for some* $k \in \mathbb{N}$*. Show that* A *can be diagonalised.*