

**Linear Algebra: Example Sheet 4 of 4**

1. The square matrices  $A$  and  $B$  over the field  $F$  are congruent if  $B = P^TAP$  for some invertible matrix  $P$  over  $F$ . Which of the following symmetric matrices are congruent to the identity matrix over  $\mathbb{R}$ , and which over  $\mathbb{C}$ ? (Which, if any, over  $\mathbb{Q}$ ?) Try to get away with the minimum calculation.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}.$$

2. Find the rank and signature of the following quadratic forms over  $\mathbb{R}$ .

$$x^2 + y^2 + z^2 - 2xz - 2yz, \quad x^2 + 2y^2 - 2z^2 - 4xy - 4yz, \quad 16xy - z^2, \quad 2xy + 2yz + 2zx.$$

If  $A$  is the matrix of the first of these (say), find a non-singular matrix  $P$  such that  $P^TAP$  is diagonal with entries  $\pm 1$ .

3. (i) Show that the function  $\psi(A, B) = \text{tr}(AB^T)$  is a symmetric positive definite bilinear form on the space  $\text{Mat}_n(\mathbb{R})$  of all  $n \times n$  real matrices. Deduce that  $|\text{tr}(AB^T)| \leq \text{tr}(AA^T)^{1/2}\text{tr}(BB^T)^{1/2}$ .  
 (ii) Show that the map  $A \mapsto \text{tr}(A^2)$  is a quadratic form on  $\text{Mat}_n(\mathbb{R})$ . Find its rank and signature.
4. Let  $\psi : V \times V \rightarrow \mathbb{C}$  be a Hermitian form on a complex vector space  $V$ .  
 (i) Find the rank and signature of  $\psi$  in the case  $V = \mathbb{C}^3$  and

$$\psi(x, x) = |x_1 + ix_2|^2 + |x_2 + ix_3|^2 + |x_3 + ix_1|^2 - |x_1 + x_2 + x_3|^2.$$

(ii) Show in general that if  $n > 2$  then  $\psi(u, v) = \frac{1}{n} \sum_{k=1}^n \zeta^{-k} \psi(u + \zeta^k v, u + \zeta^k v)$  where  $\zeta = e^{2\pi i/n}$ .

5. Show that the quadratic form  $2(x^2 + y^2 + z^2 + xy + yz + zx)$  is positive definite. Write down an orthonormal basis for the corresponding inner product on  $\mathbb{R}^3$ . Compute the basis of  $\mathbb{R}^3$  obtained by applying the Gram-Schmidt process to the standard basis with respect to this inner product.
6. Let  $W \leq V$  with  $V$  an inner product space. An endomorphism  $\pi$  of  $V$  is called an *idempotent* if  $\pi^2 = \pi$ . Show that the orthogonal projection onto  $W$  is a self-adjoint idempotent. Conversely show that any self-adjoint idempotent is orthogonal projection onto its image.
7. An endomorphism  $\alpha$  of a finite dimensional inner product space  $V$  is *positive definite* if it is self-adjoint and satisfies  $\langle \alpha(\mathbf{x}), \mathbf{x} \rangle > 0$  for all non-zero  $\mathbf{x} \in V$ .  
 (i) Prove that a positive definite endomorphism has a unique positive definite square root.  
 (ii) Let  $\alpha$  be an invertible endomorphism of  $V$  and  $\alpha^*$  its adjoint. By considering  $\alpha^*\alpha$ , show that  $\alpha$  can be factored as  $\beta\gamma$  with  $\beta$  unitary and  $\gamma$  positive definite.
8. Let  $V$  be a finite dimensional complex inner product space, and let  $\alpha$  be an endomorphism on  $V$ . Assume that  $\alpha$  is *normal*, that is,  $\alpha$  commutes with its adjoint:  $\alpha\alpha^* = \alpha^*\alpha$ . Show that  $\alpha$  and  $\alpha^*$  have a common eigenvector  $\mathbf{v}$ , and the corresponding eigenvalues are complex conjugates. Show that the subspace  $\langle \mathbf{v} \rangle^\perp$  is invariant under both  $\alpha$  and  $\alpha^*$ . Deduce that there is an orthonormal basis of eigenvectors of  $\alpha$ .
9. Find a linear transformation which simultaneously reduces the pair of real quadratic forms

$$2x^2 + 3y^2 + 3z^2 - 2yz, \quad x^2 + 3y^2 + 3z^2 + 6xy + 2yz - 6zx$$

to the forms

$$X^2 + Y^2 + Z^2, \quad \lambda X^2 + \mu Y^2 + \nu Z^2$$

for some  $\lambda, \mu, \nu \in \mathbb{R}$  (which should turn out in this example to be integers).

Does there exist a linear transformation which reduces the pair of real quadratic forms  $x^2 - y^2, \quad 2xy$  simultaneously to diagonal forms?

10. Let  $P_n$  be the  $(n + 1)$ -dimensional space of real polynomials of degree  $\leq n$ . Define

$$(f, g) = \int_{-1}^{+1} f(t)g(t)dt.$$

Show that  $(\cdot, \cdot)$  is an inner product on  $P_n$  and that the endomorphism  $\alpha : P_n \rightarrow P_n$  defined by

$$\alpha(f)(t) = (1 - t^2)f''(t) - 2tf'(t)$$

is self-adjoint. If  $f$  is an eigenvector of  $\alpha$  of degree  $k$ , what is the corresponding eigenvalue? Why must  $\alpha$  have precisely one monic eigenvector of degree  $k$  for each  $0 \leq k \leq n$ ?

Let  $s_k \in P_n$  be defined by  $s_k(t) = \frac{d^k}{dt^k}(1 - t^2)^k$ . Prove the following.

- (i) For  $i \neq j$ ,  $(s_i, s_j) = 0$ .
- (ii)  $s_0, \dots, s_n$  forms a basis for  $P_n$ .
- (iii) For all  $1 \leq k \leq n$ ,  $s_k$  spans the orthogonal complement of  $P_{k-1}$  in  $P_k$ .
- (iv)  $s_k$  is an eigenvector of  $\alpha$ .

What is the relation between the  $s_k$  and the result of applying Gram-Schmidt to the sequence  $1, x, x^2, x^3$  and so on? Explain why that is the case.

11. Let  $f_1, \dots, f_t, f_{t+1}, \dots, f_{t+u}$  be linear functionals on the finite dimensional real vector space  $V$ . Show that  $Q(\mathbf{x}) = f_1(\mathbf{x})^2 + \dots + f_t(\mathbf{x})^2 - f_{t+1}(\mathbf{x})^2 - \dots - f_{t+u}(\mathbf{x})^2$  is a quadratic form on  $V$ . Suppose  $Q$  has rank  $p + q$  and signature  $p - q$ . Show that  $p \leq t$  and  $q \leq u$ .
12. Suppose that  $\alpha$  is an orthogonal endomorphism on the finite-dimensional real inner product space  $V$ . Prove that  $V$  can be decomposed into a direct sum of mutually orthogonal  $\alpha$ -invariant subspaces of dimension 1 or 2. Determine the possible matrices of  $\alpha$  with respect to orthonormal bases in the cases where  $V$  has dimension 1 or dimension 2.
13. Let  $a_1, a_2, \dots, a_n$  be real numbers such that  $a_1 + \dots + a_n = 0$  and  $a_1^2 + \dots + a_n^2 = 1$ . What is the maximum value of  $a_1a_2 + a_2a_3 + \dots + a_{n-1}a_n + a_na_1$ ?