

## IB GROUPS, RINGS AND MODULES

Lent Term 2022

Example Sheet 1 of 4

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- (1)
  - (a) What are the orders of elements of the group  $S_4$ ? How many elements are there of each order?
  - (b) How many subgroups of order 2 are there in  $S_4$ ? Of order 3? How many cyclic subgroups are there of order 4?
  - (c) Find a non-cyclic subgroup  $V \leq S_4$  of order 4. How many such subgroups are there?
  - (d) Find a subgroup  $D \leq S_4$  of order 8. How many such subgroups are there?
- (2)
  - (a) Show that  $A_4$  has no subgroups of index 2. Exhibit a subgroup of index 3.
  - (b) Show that  $A_5$  has no subgroups of index 2, 3, or 4. Exhibit a subgroup of index 5.
  - (c) Show that  $A_5$  is generated by  $(12)(34)$  and  $(135)$ .
- (3) Calculate the size of the conjugacy class of  $(123)$  as an element of  $S_4$ , as an element of  $S_5$ , and as an element of  $S_6$ . Find in each case its centraliser. Hence calculate the size of the conjugacy class of  $(123)$  in  $A_4$ , in  $A_5$ , and in  $A_6$ .
- (4) Suppose that  $H, K \triangleleft G$  with  $H \cap K = 1$ . Show that any element of  $H$  commutes with any element of  $K$ . Hence show that  $HK \cong H \times K$ .
- (5) Let  $p$  be a prime number, and  $G$  be a non-abelian group of order  $p^3$ .
  - (a) Show that the centre  $Z(G)$  of  $G$  has order  $p$ .
  - (b) Show that if  $g \notin Z(G)$  then its centraliser  $C(g)$  has order  $p^2$ .
  - (c) Hence determine the sizes and numbers of conjugacy classes in  $G$ .
- (6)
  - (a) For  $p = 2, 3$  find a Sylow  $p$ -subgroup of  $S_4$ , and find its normaliser.
  - (b) For  $p = 2, 3, 5$  find a Sylow  $p$ -subgroup of  $A_5$ , and find its normaliser.
- (7) Show that there are no simple groups of orders 441 or 351.
- (8) Let  $p, q$ , and  $r$  be prime numbers, not necessarily distinct. Show that no group of order  $pq$  is simple. Show that no group of order  $pq^2$  is simple. Show that no group of order  $pqr$  is simple.
- (9)
  - (a) Show that any group of order 15 is cyclic.
  - (b) Show that any group of order 30 has a normal subgroup of order 15.
- (10) (Semi-direct product) Let  $N$  and  $H$  be groups, and  $\phi : H \rightarrow \text{Aut}(N)$  a homomorphism. Show that we can define a group operation on the set  $N \times H$  by

$$(n_1, h_1) \bullet (n_2, h_2) = (n_1 \cdot \phi(h_1)(n_2), h_1 \cdot h_2).$$

Show that the resulting group  $G$  contains copies of  $N$  and  $H$  as subgroups, that  $N$  is normal in  $G$ , that  $NH = G$ , and that  $N \cap H = 1$ .

By finding an element of order 3 in  $\text{Aut}(C_7)$ , construct a non-abelian group of order 21.

### Further Questions

- (11) Let  $p$  be a prime number. How many elements of order  $p$  are there in  $S_p$ ? What are their centralisers? How many Sylow  $p$ -subgroups are there? What are the orders of their normalisers? If  $q$  is another prime number which divides  $p - 1$ , show that there exists a non-abelian group of order  $pq$ .
- (12) Show that there are no simple groups of order 300 or 112.
- (13) Show that a group  $G$  of order 1001 contains normal subgroups of order 7, 11, and 13. Hence show that  $G$  is cyclic. [Hint: You may want to use Question 4.]
- (14) Let  $G$  be a simple group of order 60. Deduce that  $G \cong A_5$ , as follows. Show that  $G$  has six Sylow 5-subgroups. By considering the conjugation action on the set of Sylow 5-subgroups, show that  $G$  is isomorphic to a subgroup  $G \leq A_6$  of index 6. By considering the action of  $A_6$  on  $A_6/G$ , show that there is an automorphism of  $A_6$  taking  $G$  to  $A_5$ .
- (15) Let  $G$  be a group of order 60 which has more than one Sylow 5-subgroup. Show that  $G$  is simple.
- (16) Let  $G$  be a finite group with cyclic and non-trivial Sylow 2-subgroup. By considering the permutation representation of  $G$  on itself, show that  $G$  has a normal subgroup of index 2. [Hint: Show that a generator for the Sylow subgroup induces an odd permutation of  $G$ .]
- (17) (Frattini argument) Let  $K \triangleleft G$  and  $P$  be a Sylow  $p$ -subgroup of  $K$ . Show that any element  $g \in G$  may be written as  $g = nk$  with  $n \in N_G(P)$  and  $k \in K$ , and hence that  $G = N_G(P)K$ . [Hint: Observe that  $P$  and  $g^{-1}Pg$  are both Sylow  $p$ -subgroups of  $K$ .] Deduce that  $G/K \cong N_G(P)/N_K(P)$ .

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