GROUPS – ADDITIONAL PROBLEMS

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1. Define the *order sequence* of a finite group to be a list of the orders of its elements, written in increasing order. For example, S_3 has order sequence $(1, 2, 2, 2, 3, 3)$. If two finite groups have the same order sequence, must they be isomorphic?

2. Is there a non-cyclic group of order 55? Is there a non-cyclic group of order 65?

3. Let X be a (non-empty) set with an associative binary operation, such that for every $x \in X$ there is a unique x' such that $xx'x = x$. Prove that X is a group.

4. Which groups contain a (non-zero) even number of elements of order 2?

5. Let G be a group, $H < G$, and $g \in G$. If $gHg^{-1} \subset H$, must we have $gHg^{-1} = H$?

6. Let G be a finite group, and H a subgroup of G. Show that we can choose common representatives of the left and right cosets – that is, choose $g_1, \ldots, g_k$ such that the left cosets are $g_1H, \ldots, g_kH$ and the right cosets are $Hg_1, \ldots, Hg_k$.

7. Let G be a finite non-abelian group. Show that at most $5/8$ of the pairs of elements of G commute, and show that this bound cannot be improved.

8. Which finite groups have exactly one non-identity automorphism? Find an infinite group, other than $\mathbb{Z}$, with this property.

9. Which finite groups have the property that all non-identity elements are conjugate? Is there an infinite group with this property?

10. For which natural numbers n is there a unique group of order n?

11. Let G and H be groups such that $G \times \mathbb{Z} \cong H \times \mathbb{Z}$. Show that if G and H are abelian then they must be isomorphic. If they are non-abelian, must they be isomorphic?