GROUPS – ADDITIONAL PROBLEMS

G. Taylor

1. Which groups have exactly three subgroups?

2. Define the *order sequence* of a finite group to be a list of the orders of its elements, written in increasing order. For example, S_3 has order sequence $(1, 2, 2, 2, 3, 3)$. If two finite groups have the same order sequence, must they be isomorphic?

3. Is there a non-cyclic group of order 15? Is there a non-cyclic group of order 21?

4. Let X be a non-empty set with an associative binary operation, such that for every $x \in X$ there is a unique x' such that $xx'x = x$. Prove that X is a group.

5. Which groups contain a (non-zero) even number of elements of order 2?

6. Let G be a group, $H < G$, and $g \in G$. If $gHg^{-1} \subset H$, must we have $gHg^{-1} = H$?

7. Let G be a finite non-abelian group. Show that at most $5/8$ of the pairs of elements of G commute, and show that this bound cannot be improved.

8. Which finite groups have exactly one non-identity automorphism? Find an infinite group, other than $\mathbb{Z}$, with this property.

9. Which finite groups have the property that all non-identity elements are conjugate? Is there an infinite group with this property?

10. Write down a finite group G having a non-normal subgroup of index 5. Is there an example where G has odd order?

11. For which natural numbers n is there a unique group of order n?