

IA Groups – Example Sheet 1

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Questions marked * are more challenging.

1. Let G be any group. Show that the identity e is the only element $g \in G$ satisfying the equation $g^2 = g$.
2. Let H and K be two subgroups of a group G . Show that the intersection $H \cap K$ is a subgroup of G . Show that the union $H \cup K$ is a subgroup of G if and only if either $H \subseteq K$ or $K \subseteq H$.
3. Let $G = \mathbb{R} \setminus \{-1\}$, and let $x * y = x + y + xy$, where xy denotes the usual product of two real numbers. Show that $(G, *, 0)$ is a group. What is the inverse of 2 in this group? Solve the equation $2 * x * 5 = 6$.
4. Let G be a finite group.
 - (a) Let $g \in G$. Show from first principles that there is a positive integer n such that $g^n = e$. (The least such n is called the *order* of g .)
 - (b) Show from first principles that there is a positive integer N such that $g^N = e$ for all $g \in G$.

5. Let S be a finite non-empty set of non-zero complex numbers which is closed under multiplication. Show that S is a subset of the set $\{z \in \mathbb{C} \mid |z| = 1\}$. Show that S is a group with respect to multiplication, and deduce that S is the set of n^{th} roots of unity for some $n \in \mathbb{N}$; that is,

$$S = \{e^{2\pi i k/n} \mid k = 0, 1, \dots, n-1\}.$$

6. Show that the set of complex numbers

$$G = \{e^{\pi i t} \mid t \in \mathbb{Q}\}$$

is a group under multiplication. Show that G is infinite, but that every element of G has finite order. * Does G have an infinite, proper subgroup?

7. Let $f : G \rightarrow H$ be a group homomorphism and let $g \in G$ have finite order. Show that the order of $f(g)$ is finite and divides the order of g .
8. Show that any subgroup of a cyclic group is cyclic.
9. Let G be a group and X a subset of G .

- (a) Prove that

$$\langle X \rangle = \{x_1^{\pm 1} x_2^{\pm 1} \cdots x_n^{\pm 1} \in G \mid \text{for all } n \geq 0 \text{ and all } x_1, \dots, x_n \in X\}.$$

[The product of 0 elements of G is interpreted as the identity element.]

- (b) Now suppose that $G = \langle X \rangle$ and that $\phi, \psi : G \rightarrow K$ are both homomorphisms. Prove that, if $\phi(x) = \psi(x)$ for all $x \in X$, then $\phi = \psi$.
10. Let C_n be the cyclic group with n elements and D_{2n} the group of symmetries of the regular n -gon. If n is odd and $\theta : D_{2n} \rightarrow C_n$ is a homomorphism, show that $\theta(g) = e$ for all $g \in D_{2n}$. Can you find all homomorphisms $D_{2n} \rightarrow C_n$ if n is even? Prove that the homomorphisms $C_n \rightarrow C_m$ correspond to the elements x of C_m such that $x^n = e$.
11. Let G be a group in which every element other than the identity has order two. Show that G is abelian. Show also that, if G is finite, then the order of G is a power of 2. [*Hint: consider a minimal generating set for G .*]

12. Let G be a finite group of even order. Show from first principles that G contains an element of order two. Can a group have exactly two elements of order two?
13. Show that every isometry of $\mathbb{C}$ is either of the form $z \mapsto az + b$ or the form $z \mapsto a\bar{z} + b$ with $a, b \in \mathbb{C}$ and $|a| = 1$ in either case. * Describe the finite subgroups of the group of isometries of $\mathbb{C}$. [*Hint: First show that every finite subgroup of $\text{Isom}(\mathbb{C})$ fixes a point in $\mathbb{C}$.*]
14. Suppose that Q is a quadrilateral in $\mathbb{C}$. Show that its group of isometries $\text{Isom}(Q)$ has order at most 8. For which n is there an $\text{Isom}(Q)$ of order n ? * Which groups can arise as an $\text{Isom}(Q)$ (up to isomorphism)?