IA Groups – Example Sheet 3

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Questions marked * are more challenging. As usual, 'identify' means 'find a standard group that it is isomorphic to'.

1. Suppose $a, b \in \mathbb{Z}$ and consider $\phi : \mathbb{Z}^2 \to \mathbb{Z}$ given by $\phi(x, y) = ax + by$. Show that ϕ is a group homomorphism and describe $\text{im}(\phi)$ and $\ker(\phi)$. Draw a picture illustrating the cosets of $\ker(\phi)$ in $\mathbb{Z}^2$.

2. Show that every group of order 10 is cyclic or dihedral. * Can you extend your proof to groups of order $2p$, where p is any odd prime number?

3. Prove that
$$(\sigma p)(x_1, x_2, x_3, x_4) = p(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$$
defines an action of the group S_4 on the set of polynomials in variables x_1, x_2, x_3, x_4. Show that the stabiliser H of the polynomial $x_1x_2 + x_3x_4$ has order 8, and identify it.

4. Let p be a prime and let G be a group of order p^2. By considering the conjugation action of G on itself, show that G is abelian. Furthermore, show that there are just two groups of that order for each prime p, up to isomorphism.

5. Show that a subgroup of a group G is normal if and only if it is a union of conjugacy classes in G.

6. Let K be a subgroup of a group G. Show that K is a normal subgroup if and only if it is the kernel of some group homomorphism $\phi : G \to H$.

7. (a) Let $H \leqslant C_n$. Identify the quotient C_n/H.
   (b) Show that any subgroup $N \leqslant D_{2n}$ consisting only of rotations is normal. Identify the quotient D_{2n}/N.
   (c) Consider the subgroup
   $$\Gamma = \{m + in \mid m, n \in \mathbb{Z}\}$$
   of $\mathbb{C}$. Show that the group $\mathbb{C}/\Gamma$ is isomorphic to $S^1 \times S^1$, where S^1 is the group of complex numbers with modulus 1.

8. Suppose that G is a group in which every subgroup is normal. Must G be abelian?

9. Let G be a finite group and H a proper subgroup. Let $k = |G : H|$ and suppose that $|G|$ does not divide $k!$. By considering the action of G on G/H, show that H contains a non-trivial normal subgroup of G.

10. (a) Show that a group of order 28 has a normal subgroup of order 7.
    (b) Show that if a group G of order 28 has a normal subgroup of order 4 then G is abelian.

11. * Let G be a (not necessarily finite) group generated by a finite set X. Prove that the number of subgroups of a given index n in G is finite, and give a bound for this number in terms of n and $|X|$.

12. Write the following permutations as compositions of disjoint cycles and hence compute their orders:
    (a) (12)(1234)(12);
    (b) (123)(1234)(132);
    (c) (123)(235)(345)(45).

13. What is the largest possible order of an element of S_5? Of S_9?

14. Show that S_n is generated by each of the following sets of permutations:

   (a) $\{(j, j+1) \mid 1 \leq j < n\}$;

   (b) $\{(1, k) \mid 1 < k \leq n\}$;

   (c) $\{(12), (123 \cdots n)\}$.

15. Let $X = \mathbb{Z}/31\mathbb{Z}$, and $\sigma : X \to X$ be given by $\sigma(x + 31\mathbb{Z}) = 2x + 31\mathbb{Z}$. Show that σ is a permutation, and decompose it as a composition of disjoint cycles.

16. * Prove that S_n has a subgroup isomorphic to Q_8 if and only if $n \geq 8$.