IA Groups – Example Sheet 1

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Questions marked * are more challenging.

1. Let G be any group. Show that the identity e is the only element $g \in G$ satisfying the equation $g^2 = g$.

2. Let H and K be two subgroups of a group G. Show that the intersection $H \cap K$ is a subgroup of G. Show that the union $H \cup K$ is a subgroup of G if and only if either $H \subseteq K$ or $K \subseteq H$.

3. Let $G = \mathbb{R} \setminus \{-1\}$, and let $x * y = x + y + xy$, where xy denotes the usual product of two real numbers. Show that $(G, *, 0)$ is a group. What is the inverse of 2 in this group? Solve the equation $2 * x * 5 = 6$.

4. Let G be a finite group.

   (a) Let $g \in G$. Show from first principles that there is a positive integer n such that $g^n = e$. (The least such n is called the *order* of g.)

   (b) Show from first principles that there is a positive integer N such that $g^N = e$ for all $g \in G$.

5. Let S be a finite non-empty set of non-zero complex numbers which is closed under multiplication. Show that S is a subset of the set $\{z \in \mathbb{C} \mid |z| = 1\}$. Show that S is a group with respect to multiplication, and deduce that S is the set of n^{th} roots of unity for some $n \in \mathbb{N}$; that is,

$$S = \{e^{2\pi ik/n} \mid k = 0, 1, \ldots, n-1\}\,.$$

6. Show that the set of complex numbers

$$G = \{e^{\pi it} \mid t \in \mathbb{Q}\}$$

   is a group under multiplication. Show that G is infinite, but that every element of G has finite order. * Does G have an infinite, proper subgroup?

7. Let $f : G \to H$ be a group homomorphism and let $g \in G$ have finite order. Show that the order of $f(g)$ is finite and divides the order of g.

8. Show that any subgroup of a cyclic group is cyclic.

9. Let G be a group and X a subset of G.

   (a) Let H be a subgroup of G that contains X. Prove that $\langle X \rangle \subseteq H$.

   (b) Now suppose that $G = \langle X \rangle$ and that $\phi, \psi : G \to K$ are both homomorphisms. Prove that, if $\phi(x) = \psi(x)$ for all $x \in X$, then $\phi = \psi$.

10. Let C_n be the cyclic group with n elements and D_{2n} the group of symmetries of the regular n-gon. If n is odd and $\theta : D_{2n} \to C_n$ is a homomorphism, show that $\theta(g) = e$ for all $g \in D_{2n}$. Can you find all homomorphisms $D_{2n} \to C_n$ if n is even? Find all homomorphisms $C_n \to C_m$.

11. Let G be a group in which every element other than the identity has order two. Show that G is abelian. Show also that, if G is finite, then the order of G is a power of 2. [*Hint: consider a minimal generating set for G.*]

12. Let G be a finite group of even order. Show from first principles that G contains an element of order two. Can a group have exactly two elements of order two?

13. Show that every isometry of $\mathbb{C}$ is either of the form $z \mapsto az + b$ or the form $z \mapsto a\bar{z} + b$ with $a, b \in \mathbb{C}$ and $|a| = 1$ in either case. * Describe the finite subgroups of the group of isometries of $\mathbb{C}$. [*Hint: First show that every finite subgroup of* $\mathrm{Isom}(\mathbb{C})$ *fixes a point in* $\mathbb{C}$.]

14. Suppose that Q is a quadrilateral in $\mathbb{C}$. Show that its group of isometries $\mathrm{Isom}(Q)$ has order at most 8. For which n is there an $\mathrm{Isom}(Q)$ of order n? * Which groups can arise as an $\mathrm{Isom}(Q)$ (up to isomorphism)?