## Permutation groups

1. Show that if $H$ is a subgroup of $S_n$ containing an odd permutation then exactly half of the elements of $H$ are odd.

2. (a) Show that $A_4$ has no subgroup of order 6.

   (b) Show that $S_4$ has a subgroup of order $d$ for each $d$ dividing 24. For which $d$ does $S_4$ have two non-isomorphic subgroups of order $d$?

3. Find the centre of each of $S_n$ and $A_n$, for all $n$.

4. Let $\sigma \in A_n$. Show that the conjugacy class of $\sigma$ in $A_n$ is half of that in $S_n$ if and only if the cycles (including singletons) in the disjoint cycle decomposition of $\sigma$ have distinct odd lengths.

5. Determine the sizes of the conjugacy classes in $A_6$. Deduce that $A_6$ is a simple group.

6. By using an action on left cosets, show that $A_5$ has no subgroup of index 2, 3 or 4, and that any subgroup of index 5 is isomorphic to $A_4$.

## Matrix groups

7. Let $G$ be the set of all $3 \times 3$ real matrices of determinant 1 of the form
$$\begin{pmatrix} a & 0 & 0 \\ b & w & x \\ c & y & z \end{pmatrix}.$$
Show that $G$ is a subgroup of $GL_3(\mathbb{R})$. Construct a surjective homomorphism from $G$ to $GL_2(\mathbb{R})$, and find its kernel.

8. Let $G$ be the set of all $3 \times 3$ real matrices of the form
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$
Show that $G$ is a subgroup of $GL_3(\mathbb{R})$. Let $H \subset G$ be the subset of those matrices with $a = c = 0$. Show that $H$ is a normal subgroup of $G$, and identify the quotient group $G/H$.

9. Show that the only normal subgroup of $O(2)$ containing a reflection is $O(2)$ itself.

10. (a) Find a surjective homomorphism from $O(3)$ to $C_2$ and another from $O(3)$ to $SO(3)$.

    (b) Prove that $O(3)$ is isomorphic to $SO(3) \times C_2$.

    (c) Is $O(4)$ isomorphic to $SO(4) \times C_2$?

11. For $A \in M_{n\times n}(\mathbb{C})$ with entries $a_{ij}$, let $A^\dagger \in M_{n\times n}(\mathbb{C})$ have entries $\overline{a_{ji}}$. The matrix $A$ is called *unitary* if $AA^\dagger = I_n$. Show that the set $U(n)$ of unitary matrices is a subgroup of $GL_n(\mathbb{C})$. Show that $SU(n) = \{A \in U(n) : \det A = 1\}$ is a normal subgroup of $U(n)$ and that $U(n)/SU(n)$ is isomorphic to $S^1$.

12. Let $SL_2(\mathbb{R})$ act on $\mathbb{C}_\infty$ via Möbius transformations. Find the orbit and stabiliser of $i$ and $\infty$. By considering the orbit of $i$ under the action of the stabiliser of $\infty$, show that every $g \in SL_2(\mathbb{R})$ may be written as $g = hk$ with $h$ upper-triangular and $k \in SO(2)$. In how many ways can this be done?

13. Let $p$ be prime, let $G = GL_2(\mathbb{Z}_p)$ be the group of invertible matrices modulo $p$, and let $X = \mathbb{Z}_p^2$ be the set of vectors of length 2 with entries in $\mathbb{Z}_p$.

    (i) Show that $G$ acts on $X$ by

    $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

    Find the orbit and stabiliser of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and hence find the order of $G$.

    (ii) Let $g \in G$ have order $p$. Show that $g$ fixes some non-zero vector in $X$, and deduce that $g$ is conjugate in $G$ to

    $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

## Optional extras

14. Let $G$ be a finite non-trivial subgroup of $SO(3)$. Let $X$ be the set of points on the unit sphere in $\mathbb{R}^3$ which are fixed by at least one non-trivial rotation in $G$. Show that $G$ acts on $X$ and that there are either two or three orbits.

    Identify $G$ in the case when there are two orbits. When there are three orbits, what are their possible sizes?

15. Which of the following groups can occur as $G/Z(G)$ for some group $G$: $D_6$, $C_7$, $Q_8$ ?

16. Does $GL_2(\mathbb{R})$ have a subgroup isomorphic to $Q_8$?