1. Prove that D_{12} is isomorphic to $D_6 \times C_2$. Is D_{16} isomorphic to $D_8 \times C_2$?

2. Show that any group of order 10 is either cyclic or dihedral. Can you extend your proof to groups of order $2p$, where p is any odd prime number?

3. Let p be a prime and let G be a group of order p^2. By considering the conjugation action of G on itself, show that G is abelian. Show that there are, up to isomorphism, just two groups of order p^2 for each prime p.

4. Let K be a subgroup of a group G. Show that:

   (a) K is normal if and only if it is a union of conjugacy classes
   (b) K is normal if and only if it is the kernel of some homomorphism of G.

5. (a) Let $H \leqslant C_n$. Identify the quotient C_n/H. (That is, which standard group is it isomorphic to?)
   (b) Show that any subgroup $N \leqslant D_{2n}$ consisting only of rotations is normal. Identify the quotient D_{2n}/N.

6. Let G be a group. If H is a normal subgroup of G, and K is a normal subgroup of H, must K be a normal subgroup of G?

7. Let G be a finite group and H a proper subgroup. Let $k = |G : H|$ and suppose that $|G|$ does not divide $k!$. By considering the action of G on the cosets of H, show that H contains a non-trivial normal subgroup of G.

8. Let G be a group of order 28. Show that G has a normal subgroup of order 7. Show that if G also has a normal subgroup of order 4 then G is abelian.

9. Write the following as products of disjoint cycles, and hence find their order and sign.

   (i) (12)(1234)(12)
   (ii) (123)(1234)(132)
   (iii) (123)(235)(345)(45)

10. What is the largest possible order of an element of S_5? And of S_9?

11. (a) Show that S_n is generated by each of the following sets of permutations:
    (i) $\{(j, j+1) : 1 \leqslant j < n\}$
    (ii) $\{(1, k) : 1 < k \leqslant n\}$
    (iii) $\{(12), (123 \cdots n)\}$.
    (b) What subgroup of S_n does the set of n-cycles generate?

12. Prove that S_n has a subgroup isomorphic to Q_8 if and only if $n \geqslant 8$.

*13. Let G be a (not necessarily finite) group generated by a finite set X. Prove that the number of subgroups of a given index n in G is finite, and give a bound for this number in terms of n and $|X|$.

*14. Let G be a group of order $p^a m$, where p is prime and m is coprime to p. By considering an action on the subsets of G of size p^a, prove that G has a subgroup of order p^a.