Permutation groups

1. Show that if H is a subgroup of S_n containing an odd permutation then exactly half of the elements of H are odd.

2. (a) Show that A_4 has no subgroup of order 6.

   (b) Show that S_4 has a subgroup of order d for each d dividing 24. For which d does S_4 have two non-isomorphic subgroups of order d?

3. Find the centre of each of S_n and A_n, for all n.

4. Let $\sigma \in A_n$. Show that the conjugacy class of σ in A_n is half of that in S_n if and only if the cycles (including singletons) in the disjoint cycle decomposition of σ have distinct odd lengths.

5. Determine the sizes of the conjugacy classes in A_6. Deduce that A_6 is a simple group.

6. By using an action on left cosets, show that A_5 has no subgroup of index 2, 3 or 4, and that any subgroup of index 5 is isomorphic to A_4.

Matrix groups

7. Let G be the set of all 3×3 real matrices of determinant 1 of the form
$$\begin{pmatrix} a & 0 & 0 \\ b & w & x \\ c & y & z \end{pmatrix}.$$
Show that G is a subgroup of $GL_3(\mathbb{R})$. Construct a surjective homomorphism from G to $GL_2(\mathbb{R})$, and find its kernel.

8. Let G be the set of all 3×3 real matrices of the form
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$
Show that G is a subgroup of $GL_3(\mathbb{R})$. Let $H \subset G$ be the subset of those matrices with $a = c = 0$. Show that H is a normal subgroup of G, and identify the quotient group G/H.

9. Show that the only normal subgroup of $O(2)$ containing a reflection is $O(2)$ itself.

10. (a) Find a surjective homomorphism from $O(3)$ to C_2 and another from $O(3)$ to $SO(3)$.

    (b) Prove that $O(3)$ is isomorphic to $SO(3) \times C_2$.

    (c) Is $O(4)$ isomorphic to $SO(4) \times C_2$?

11. For $A \in M_{n\times n}(\mathbb{C})$ with entries a_{ij}, let $A^\dagger \in M_{n\times n}(\mathbb{C})$ have entries $\overline{a_{ji}}$. The matrix A is called *unitary* if $AA^\dagger = I_n$. Show that the set $U(n)$ of unitary matrices is a subgroup of $GL_n(\mathbb{C})$. Show that $SU(n) = \{A \in U(n) : \det A = 1\}$ is a normal subgroup of $U(n)$ and that $U(n)/SU(n)$ is isomorphic to S^1.

12. Let $SL_2(\mathbb{R})$ act on $\mathbb{C}_\infty$ via Möbius transformations. Find the orbit and stabiliser of i and ∞. By considering the orbit of i under the action of the stabiliser of ∞, show that every $g \in SL_2(\mathbb{R})$ may be written as $g = hk$ with h upper-triangular and $k \in SO(2)$. In how many ways can this be done?

13. Let p be prime, let $G = GL_2(\mathbb{Z}_p)$ be the group of invertible matrices modulo p, and let $X = \mathbb{Z}_p^2$ be the set of vectors of length 2 with entries in $\mathbb{Z}_p$.

    (i) Show that G acts on X by
    $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$
    Find the orbit and stabiliser of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and hence find the order of G.

    (ii) Let $g \in G$ have order p. Show that g fixes some non-zero vector in X, and deduce that g is conjugate in G to
    $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Optional extras

14. Let G be a finite non-trivial subgroup of $SO(3)$. Let X be the set of points on the unit sphere in $\mathbb{R}^3$ which are fixed by at least one non-trivial rotation in G. Show that G acts on X and that there are either two or three orbits.

    Identify G in the case when there are two orbits. When there are three orbits, what are their possible sizes?

15. Which of the following groups can occur as $G/Z(G)$ for some group G: D_6, C_7, Q_8 ?

16. Does $GL_2(\mathbb{R})$ have a subgroup isomorphic to Q_8?