1. Let G be generated by a, b such that $aba^{-1} = b^2$ and $bab^{-1} = a^2$. What is the order of G?

2. Show that the dihedral group D_{2n} may be generated by two elements of order 2.
   Show that a finite group generated by two elements of order 2 is isomorphic to some (possibly degenerate) dihedral group D_{2n}.
   Can D_{2n} be generated by a set containing no elements of order 2?

3. Show that no group contains precisely two elements of order 2.

4. (a) Prove that A_n is generated by its 3-cycles.
   (b) Prove that A_n is generated by (123) and $(123\ldots n)$ if n is odd, and by (123) and $(234\ldots n)$ if n is even.
   (c) What subgroup of S_n does the set of n-cycles generate?

5. (a) Show that any infinite group has infinitely many distinct subgroups.
   (b) Give an example of an infinite group in which every element has finite order.
   (c) Give an example of an infinite group in which every (proper) subgroup has finite order.

6. (a) Let the group G act on the set X. For $g \in G$, define $\text{fix}(g) = \{x \in X : g(x) = x\}$.
   By counting the set $\{(g, x) \in G \times X : g(x) = x\}$ in two ways, show that the average fix size $\frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|$ equals the number of orbits of the action.
   Deduce that if G acts transitively and $|X| > 1$ then some $g \in G$ has no fixed point.
   (b) In how many distinct ways can the faces of a cube be coloured using at most three colours? What about a dodecahedron? (We regard as equivalent two colourings that can obtained from each other by a rotation.)

7. Show that if a group of order 28 has a normal subgroup of order 4 then the group is abelian.

8. Let $SL_2(\mathbb{R})$ act on C_∞ via Möbius maps. Find the orbit and stabiliser of i and ∞. By considering the orbit of i under the action of the stabiliser of ∞, show that every $g \in SL_2(\mathbb{R})$ may be written as $g = hk$ with h upper-triangular and $k \in SO_2$.

9. (a) Construct a Möbius map that maps $\{z \in \mathbb{C} : |z - 1| < 1\}$ onto $\{z \in \mathbb{C} : |z| > 2\}$.
   (b) Construct a Möbius map that maps the strip $\{z \in \mathbb{C} : 0 < \text{Im}(z) < 1\}$ onto the region between the circles $|z - 1| = 1$ and $|z - 2| = 2$.

   Does there exist a Möbius map that maps either region in (a) onto either region in (b)?

10. Use cross-ratios to prove Ptolemy's theorem: 'for any quadrilateral whose vertices lie on a circle, the product of the lengths of the diagonals equals the sum of the products of the lengths of pairs of opposite sides'.

11. Let G be the set of integers modulo 2^n with operation
    $$x * y = 4xy + x(-1)^y + y(-1)^x \pmod{2^n} .$$
    Show that G is a cyclic group.

12. Let Q_8 be the group of quaternions.
    (a) Prove that no group G satisfies $G/Z \cong Q_8$, where Z is the centre of G.
    (b) Prove that S_n has a subgroup isomorphic to Q_8 iff $n \geqslant 8$.
    (c) Does $GL_2(\mathbb{R})$ have a subgroup isomorphic to Q_8?

13. Show that any subgroup of A_5 of order 12 is isomorphic to A_4. Show that there are precisely five such subgroups.