Part IA Groups // Example Sheet 1

1. Let G be any group. Show that the identity e is the unique solution of the equation $x^2 = x$ in G.

2. Let H_1 and H_2 be two subgroups of a group G. Show that the intersection $H_1 \cap H_2$ is a subgroup of G. Show that the union $H_1 \cup H_2$ is a subgroup of G if and only if one of the H's contains the other.

3. Let $G = \{x \in \mathbb{R} \mid x \neq -1\}$, and let $x * y = x + y + xy$, where xy denotes the usual product of two real numbers. Show that $(G, *, 0)$ is a group. What is the inverse 2^{-1} of 2 in this group? Solve the equation $2 * x * 5 = 6$.

4. Let G be a finite group. Show that every element of G has finite order. Show that there exists a positive integer N such that for all $g \in G$ we have $g^N = e$.

5. Show that the set G of complex numbers of the form $\exp(i\pi t)$ with t rational is a group under multiplication (with identity 1). Show that G is infinite, but that every element a of G has finite order.

6. Let $f : G \to H$ be a group homomorphism, and $a \in G$ have finite order. Show that the order of $f(a)$ is finite and divides the order of a.

7. Let C_n be the cyclic group with n elements and D_{2n} the group of symmetries of the regular n-gon. If n is odd and $\theta \colon D_{2n} \to C_n$ is a homomorphism, show that $\theta(g) = e$ for all $g \in D_{2n}$. Can you find all homomorphisms $D_{2n} \to C_n$ if n is even? Find all homomorphisms $C_n \to C_m$.

8. Show that any subgroup of a cyclic group is cyclic.

9. Show that the set $\{1, 3, 5, 7\}$ forms a group under multiplication modulo 8. Is it isomorphic to $C_2 \times C_2$ or C_4?

10. Let G be a group in which every element other than the identity has order two. Show that G is abelian. *Show also that if G is finite, then the order of G is a power of 2.

11. Let G be a finite group of even order. Show that G contains an element of order two.

12. Show that every isometry of $\mathbb{C}$ is either of the form $z \mapsto az + b$ or the form $z \mapsto a\bar{z} + b$ with $a, b \in \mathbb{C}$ and $|a| = 1$ in either case. *Describe the finite subgroups of the group of isometries of $\mathbb{C}$.

13. Show that $t * (x, y) := (e^t x, e^{-t} y)$ defines an action of the group $(\mathbb{R}, +, 0)$ on the set $\mathbb{R}^2$. What are the orbits and stabilisers of this action? There is a differential equation that is satisfied by each of the orbits. What is it?

14. Suppose that Q is a quadrilateral in $\mathbb{R}^2$. Show that its group of symmetries $G(Q)$ has order at most 8. For which n is there a $G(Q)$ of order n? *Which groups can arise as a $G(Q)$ (up to isomorphism)?

15. Let $S^1 := \{t \in \mathbb{C} \ s.t.\ |t| = 1\}$, which is a group under multiplication, and let
$$S^3 = \{(w_1, w_2) \in \mathbb{C}^2 \ s.t.\ |w_1|^2 + |w_2|^2 = 1\}.$$
Show that $(t_1, t_2) * (w_1, w_2) := (t_1 w_1, t_2 w_2)$ defines an action of the group $S^1 \times S^1$ on the set S^3. Describe the orbits of this action and find all stabilisers.

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