Suppose that f is analytic on the punctured disc $\{z \in \mathbb{C} \mid 0 < |z-a| < r\}$. Then it has a *Laurent expansion* $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$, valid for $0 < |z-a| < r$. The coefficients c_n are unique. Because f is analytic on a punctured disc about a, we say that f has an *isolated singularity* at a. The nature of this singularity is determined by the coefficients c_n. We have three cases:

1. If $c_n = 0$ for all $n < 0$ then f has a *removable singularity* at a. f may be extended to the full disc by defining $f(a) = c_0$. The resulting function $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$, $0 \leq |z-a| < r$, is analytic on the disc (of course, f may already be analytic on the disc). For example
$$f(z) \;=\; \frac{\sin z}{z} \;=\; 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$
for $z \neq 0$, can be extended to 0 by defining $f(0) = 1$.

2. If there is $n < 0$ such that $c_n \neq 0$ and $c_m = 0$ for all $m < n$, then f has a pole at a, of order $-n$. It is important in applications to determine the order of the pole (so that, for example, residues may be calculated properly). While the coefficients in a Taylor expansion $g(z) = \sum_{n=0}^{\infty} d_n (z-a)^n$ can be found easily in principle (by repeated differentiation, we get $d_n = g^{(n)}(a)/n!$), Laurent coefficients can be more elusive.

   Sometimes the coefficients are easy to determine. For example
$$\frac{\sin z}{z^3} \;=\; \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \dots$$
   for $z \neq 0$, so it is clear that $\sin z/z^3$ has a pole of order 2 at the origin. But in general, things are less clear. Fortunately the following result can be applied to a lot of functions. We say that f, analytic on the disc $|z-a| < r$, has a *zero of order n at a* if $f^{(k)}(a) = 0$ for $k < n$ and $f^{(n)}(a) \neq 0$; equivalently, if f has a Taylor expansion $f(z) = \sum_{k=n}^{\infty} d_k (z-a)^k$, with $d_n \neq 0$. Note that the orders of zeros are, in principle, easy to calculate by repeated differentiation.

   Proposition 1 *Let f, g be analytic on the disc $|z-a| < r$, with zeros of orders n, m at a respectively. If $n < m$ then f/g (which is analytic for $0 < |z-a| < r$) has a pole of order $m-n$ at a.*

   Proof. Omitted to keep these notes short, but quite accessible. □

   Example 2 $\tan z$ has a pole of order 1 at $\frac{\pi}{2}$. Indeed, $\sin\frac{\pi}{2} = 1$, so sin has a zero of order 0 at $\frac{\pi}{2}$, i.e. no zero. Meanwhile, $\cos\frac{\pi}{2} = 0$ and $\frac{\mathrm{d}}{\mathrm{d}z}\cos z\big|_{z=\frac{\pi}{2}} = -1$, so cos has a zero of order 1 at $\frac{\pi}{2}$. Therefore, tan has a pole of order 1 at $\frac{\pi}{2}$.

   Sometimes a function cannot be easily written as f/g with f, g analytic for $|z-a| < r$. In this case, it may be necessary to apply a brute force expansion to calculate the order of the pole.

3. If (1) and (2) do not hold then, for all $n < 0$, we can find $m \leq n$ such that $c_m \neq 0$. In this case, f has an (isolated) *essential singularity* at a. The behaviour of f as $z \to a$ is, in this case, extremely wild. For example,

$$e^{\frac{1}{z}} \;=\; 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots$$

for $z \neq 0$, has an essential singularity at the origin. As an example of the wildness, we have the following theorem.

Theorem 3 (Casorati-Weierstrass) *Let f have an isolated, essential singularity at a. Then, given any $w \in \mathbb{C}$, there is a sequence $z_n \to a$ such that $f(z_n) \to w$.*

The proof of this result is well within the scope of the Complex Analysis course.

Finally, we remark that there is a convenient way of determining the nature of the singularity at a by finding the limit of $f(z)$ as $z \to a$.

Proposition 4 *Let f be analytic on the punctured disc $0 < |z - a| < r$. Then*

1. *f has a removable singularity at a if and only if $\lim_{z\to a} f(z) = w$ for some $w \in \mathbb{C}$;*
2. *f has a pole at a if and only if $\lim_{z\to a} f(z) = \infty$;*
3. *f has an essential singularity at a if and only if $\lim_{z\to a} f(z)$ does not exist.*

Proof. Exercise. □

Example 5 Consider $e^{\frac{1}{z}}$, analytic on $\mathbb{C}\backslash\{0\}$. Since $e^{2ni\pi} = 1$ and $e^{(2n+1)i\pi} = -1$ for all $n \in \mathbb{N}$, if we set $z_n = 1/2ni\pi$ and $w_n = 1/(2n+1)i\pi$ then $z_n, w_n \to 0$ and $\lim e^{\frac{1}{z_n}} = 1$ and $\lim e^{\frac{1}{w_n}} = -1$. Hence $\lim_{z\to 0} e^{\frac{1}{z}}$ does not exist. Thus $e^{\frac{1}{z}}$ has an essential singularity at the origin.

In fact, it is easy to see the Casorati-Weierstrass Theorem (Theorem 3) in action in Example 5. Indeed, take any $w \in \mathbb{C}$. If $w = 0$, set $z_n = -\frac{1}{n}$. If $w \neq 0$ then (by considering log), there is $a \in \mathbb{C}$ such that $e^a = w$. Set $z_n = (a + 2ni\pi)^{-1}$. In both cases, we have $\lim e^{\frac{1}{z_n}} = w$.