Lemma 1 (Jordan) *Let f be continuous for large $|z|$, and assume that $f(z) \to 0$ as $z \to \infty$. Then, provided $\lambda > 0$, we have*

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z \;=\; 0$$

where Γ_R denotes the semicircular contour $\theta \mapsto R\mathrm{e}^{\mathrm{i}\theta}$, $0 \le \theta \le \pi$.

Proof. Let $\varepsilon > 0$. We show that there exists $K > 0$ such that $R \ge K$ implies

$$\Big| \int_{\Gamma_R} f(z)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z \Big| \;\le\; \varepsilon.$$

This will prove the lemma. Because $f(z) \to 0$ as $z \to \infty$, there exists $K > 0$ such that $|z| \ge K$ implies $|f(z)| \le \varepsilon\lambda/\pi$. We also note Jordan's inequality

$$\sin\theta \ge \frac{2\theta}{\pi} \qquad \text{for } 0 \le \theta \le \tfrac{\pi}{2}. \tag{1}$$

Now, for $R \ge K$, we have

$$\begin{aligned}
\Big| \int_{\Gamma_R} f(z)\mathrm{e}^{\mathrm{i}\lambda z}\,\mathrm{d}z \Big| &\le \int_{\Gamma_R} |f(z)\mathrm{e}^{\mathrm{i}\lambda z}|\,|\mathrm{d}z| \\
&= \int_{\Gamma_R} |f(z)|\mathrm{e}^{\mathrm{Re}(\mathrm{i}\lambda z)}\,|\mathrm{d}z| \\
&\le \int_0^{\pi} \frac{\lambda\varepsilon}{\pi}\mathrm{e}^{-\lambda R\sin\theta} R\,\mathrm{d}\theta && z = R\mathrm{e}^{\mathrm{i}\theta} \\
&= \frac{2\lambda R\varepsilon}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{e}^{-\lambda R\sin\theta}\,\mathrm{d}\theta && \text{by symmetry} \\
&\le \frac{2\lambda R\varepsilon}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{e}^{-2\lambda R\theta/\pi}\,\mathrm{d}\theta && \text{by (1) and } \lambda > 0 \\
&= \frac{2R\lambda\varepsilon}{\pi}\Big[\frac{-\pi}{2\lambda R}\mathrm{e}^{-2\lambda R\theta/\pi}\Big]_0^{\frac{\pi}{2}} \\
&= \varepsilon(1 - \mathrm{e}^{-\lambda R}) \\
&\le \varepsilon.
\end{aligned}$$

□