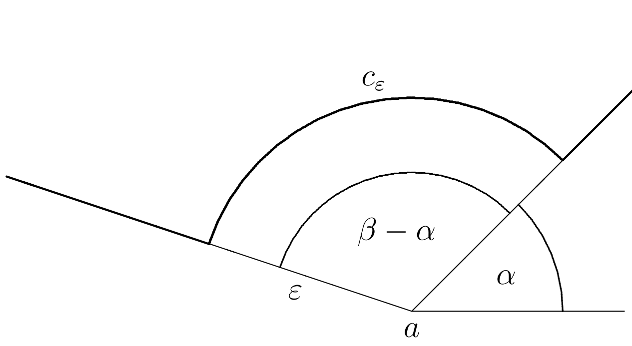
Often in contour integration the contour we would like to use passes through a simple pole at, say a. Integrating through a pole is illegal, so to avoid doing so we indent the contour by inserting a circular arc c_ε of radius $\varepsilon > 0$ and subtending an angle $\beta - \alpha$, and then taking the limit as $\varepsilon \to 0$.



The following lemma provides a very simple and useful formula for the limit of the integral of f along the arc c_ε as $\varepsilon \to 0$.

Lemma 1 *Let f have a simple pole at a with residue* $\mathrm{res}(f;a)$. *Then*

$$\lim_{\varepsilon\to 0}\int_{c_\varepsilon} f(z)\,\mathrm{d}z \;=\; (\beta-\alpha)\mathrm{i}\,\mathrm{res}(f;a)$$

where c_ε denotes the circular arc $\theta \mapsto a + \varepsilon e^{i\theta}$, $\alpha \le \theta \le \beta$.

Proof. Let $\lambda = \mathrm{res}(f;a)$. Since f has a simple pole at a, by considering the Laurent expansion of f about a, there is $r > 0$ and an analytic function g in the region $|z-a| < r$, such that

$$f(z) \;=\; \frac{\lambda}{z-a} + g(z)$$

for $0 < |z-a| < r$. By continuity of g at a, we can choose r small enough so that g is bounded by some M for $|z-a| < r$. Now, for $0 < \varepsilon < r$, we have

$$\begin{aligned}\int_{c_\varepsilon} f(z)\,\mathrm{d}z &= \lambda\int_{c_\varepsilon}\frac{\mathrm{i}}{z-a}\,\mathrm{d}z + \int_{c_\varepsilon} g(z)\,\mathrm{d}z \\ &= \lambda\int_\alpha^\beta \mathrm{i}\,\mathrm{d}\theta + \int_{c_\varepsilon} g(z)\,\mathrm{d}z \qquad z = a+\varepsilon e^{i\theta} \\ &\to \lambda(\beta-\alpha)\mathrm{i}\end{aligned}$$

as $\varepsilon \to 0$. The last integral tends to zero because

$$\left|\int_{c_\varepsilon} g(z)\,\mathrm{d}z\right| \;\le\; M \times (\text{length of } c_\varepsilon) \;=\; M(\beta-\alpha)\varepsilon \;\to\; 0.$$

□

This lemma only works if the pole at a is *simple* (locate where the proof above fails if the pole at a is not simple). If you find yourself trying to indent a contour at a non-simple pole, choose a different contour.