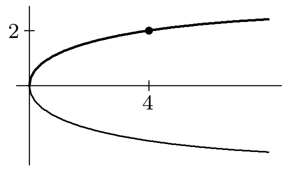
# Branches

We don't like multi-valued functions. Some people might say that such things aren't really functions at all. But these things do happen, and we have to deal with them.

To begin, let's consider the square root function on the positive reals. There are two possible values for each $x$, and we want to pick just one of them. That is, we will define a function $f$ such that for every $x$ we have $f(x)^2 = x$. For each $x$, there are two choices for $f(x)$, namely $\pm\sqrt{x}$. We could choose arbitrarily: for example, let $f(x) = +\sqrt{x}$ for $x \in \mathbb{Q}$ and $f(x) = -\sqrt{x}$ for $x \notin \mathbb{Q}$. While that might be fun as a set theory example, it's not much help for analysis – we want $f$ to be continuous.

So we'll impose continuity as a condition. And let's suppose that we decide to have $f(4) = +2$.




Thinking in terms of $\epsilon$-$\delta$ stuff, we see that for $x$ close to 4, we must have $f(x)$ close to 2. And it's not hard to see that our choice at $x = 4$ propagates outwards, forcing $f(x) = +\sqrt{x}$ for all $x$. We can never jump to a negative value, as that jump would cause discontinuity.

Similarly, if we had decided to set $f(4) = -2$, we would force $f(x) = -\sqrt{x}$ for all $x$. So while there are lots of discontinuous functions we can find, there are only two continuous ones: $f(x) = +\sqrt{x}$ for all $x$, and $f(x) = -\sqrt{x}$ for all $x$.

Let's now move to $\mathbb{C}$.

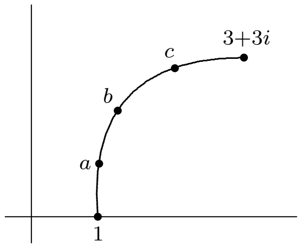
Consider $\log z$. Writing $z = re^{i\theta}$, we have $\log z = \log r + i\theta$, where by $\log r$ we mean the ordinary real logarithm function. However, the argument of a point can take many values: increasing it by $2\pi$ leaves us at the same point. For example:

$$\begin{aligned} \arg 1 &= \ldots,\ -2\pi,\ 0,\ 2\pi,\ 4\pi,\ \ldots \\ \log 1 &= \ldots,\ -2\pi i,\ 0,\ 2\pi i,\ 4\pi i,\ \ldots \end{aligned}$$

$$\begin{aligned} \arg(3+3i) &= \ldots,\ -\tfrac{7}{4}\pi,\ \tfrac{1}{4}\pi,\ \tfrac{9}{4}\pi,\ \ldots \\ \log(3+3i) &= \ldots,\ \log(3\sqrt{2}) - \tfrac{7}{4}\pi i,\ \log(3\sqrt{2}) + \tfrac{1}{4}\pi i,\ \log(3\sqrt{2}) + \tfrac{9}{4}\pi i,\ \ldots \end{aligned}$$

We will choose a particular value of the argument for each point in the plane. This uniqueness will then make the logarithm single-valued. And as with our real example above, we impose the constraint that what we define must give us a continuous function.

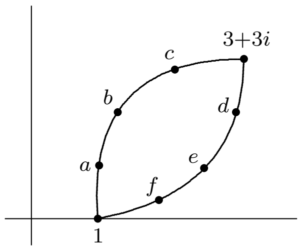
Suppose that we decide that $\arg 1 = 0$. The choices for the arguments at points close to 1 are "close to 0", "about $2\pi$ away from 0", "about $4\pi$ away from 0", etc. For continuity, we're surely going to have to pick those "close to 0". Let's start at 1 and go for a walk to $3 + 3i$.




By continuity, the choice of 0 at 1 forces the choices nearby. This propagates along our path: the choice at 1 forces the choice at $a$, which forces the choice at $b$, which forces the choice at $c$, which forces the choice at $3 + 3i$. We end up having no choice but to define $\arg(3 + 3i)$ to be $\frac{\pi}{4}$. We see that continuity makes our initial decision have consequences which propagate outwards across the plane.

Suppose we now continue on from $3+3i$, heading back to 1. If we simply retrace our steps, then it's no surprise that that at $c$, the value we're forced to take this time is what we had there last time – "picking close values" simply undoes the previous choices, giving us no choice on the way back.

But we could return by a different route:

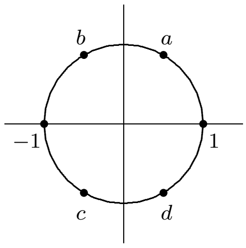



As before, by continuity, the value of $\frac{\pi}{4}$ at $3 + 3i$ forces the choice at $d$, which forces the choice at $e$, which forces the choice at $f$, ... which forces the choice at 1.

Ah, now we might be in trouble. What happens if the value we are now being forced to take at 1 doesn't equal the value we started with initially? This would be bad, as we would no longer be certain what to call the argument of 1.

But we're okay here. We can see from the picture that as we travelled, the arguments all stayed safely in the range $(0, \frac{\pi}{2})$, and so we never got near the other possible values for $\arg 1$. So our return journey from $f$ to 1 returns us happily to $\arg 1 = 0$, and all is well.

Let's go for a different walk, starting at 1 and travelling anticlockwise around the origin.

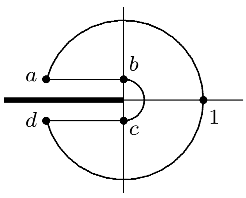



As before, the initial choice of $\arg 1 = 0$ forces the choice at $a$, which forces the choice at $b$, …, which forces the choice at $d$, which forces the choice at 1. However, this time, the argument got a bit bigger on the way from 1 to $a$, then a bit bigger from $a$ to $b$, …, then a bit bigger from $d$ to 1. Oops. On our return to 1, we're trying to force $\arg 1 = 2\pi$.

This is bad, and we're no longer sure what to call $\arg 1$. If we had started at $-1$ and headed towards 1, then we wouldn't be sure where we'd end up, as the different routes give different answers. The problem is that when we circle the origin, the argument goes up by $2\pi$.

What are the possible resolutions? We could accept multi-valuedness, but we really don't want to do that. We could drop continuity, but this is similarly bad. Our fix is to remove part of the complex plane, providing a barrier that prevents us from ever circling the origin.

For example, let us remove the negative real axis from the plane, and look at the "cut plane" $\mathbb{C} \setminus \{x + iy : x \leqslant 0, y = 0\}$. We can't make our bad walk – the closest we can manage now is the following:




But this is okay. As we travel from 1 to $a$, the argument increases from 0 to nearly $\pi$. From $a$ to $b$, it decreases to $\frac{\pi}{2}$. From $b$ to $c$, it whooshes down to $-\frac{\pi}{2}$. From $c$ to $d$, it decreases to nearly $-\pi$. And from $d$ back to 1, it increases back to 0.

Since we can no longer circle 0, the available range of arguments we can access is $(-\pi, \pi)$. So we can no longer get to the other options, and the argument is now a single-valued function.

This makes functions which use the argument (use it sensibly, anyway) single-valued as well. For example, $\log z$ is now single-valued: $\log z = \log |z| + i \arg z$.

It makes a function like $z^{1/2}$ single-valued. We have $z^{1/2} = e^{\frac{1}{2}\log z} = \sqrt{|z|}e^{\frac{1}{2}i \arg z}$. Before we cut the plane, we could have started with $z = re^{i\theta}$, giving $z^{1/2} = \sqrt{r}e^{\frac{1}{2}i\theta}$, and then circled the origin and returned to $z = re^{i(\theta+2\pi)}$, giving $z^{1/2} = \sqrt{r}e^{\frac{1}{2}i(\theta+2\pi)} = \sqrt{r}e^{\frac{1}{2}i\theta}e^{i\pi} = -\sqrt{r}e^{\frac{1}{2}i\theta}$. We have ambiguity about which square root to take. But cutting the plane prevents this, giving us a single-valued square root function.

Okay, so we can now make single-valued functions out of multi-valued functions. But which of the many options do we end up with? In the discussion above, we made some defining choices: we specified the value of the argument at 1, we specified that this argument was 0, and we specified that we lose the negative real axis. How would choosing these differently affect things?

Let's stick with the same cut – that is, we'll still look at the region $\mathbb{C} \setminus \{x + iy : x \leqslant 0, y = 0\}$. Before, we chose the argument at 1 to be 0. What if we choose the value somewhere else? We could insist that $\arg(3 + 3i) = \frac{\pi}{4}$ and see what propagates out from there. This will end up giving us the same selection of values as before: the same walk as before back to 1 will force the argument there to be 0, after all.

So we see that given a cut, to specify the range of the argument it's enough to specify the value at any single point, as then the whole function is determined by continuity. So this doesn't change things.

Again, let's stick with the same cut. As before, we'll specify the value of the argument at 1, but this time we'll set it to be $2\pi$. This choice propagates out like before, and by continuity we find that every point's argument has increased by $2\pi$. We get a new copy of the function, just like if we'd followed the negative choice in the square root function on $\mathbb{R}$ at the start.

Such a choice affects functions which involve the argument. The most obvious is that the value of $\log z$ has increased by $2\pi i$. How about the square root function?

Before, we had $z^{1/2} = \sqrt{r}e^{\frac{1}{2}i\theta}$, with $\theta \in (-\pi, \pi)$. In particular, for real $z$ we had $z^{1/2} = \sqrt{r}$, the positive square root. Now, we have $\theta \in (\pi, 3\pi)$, so at the real number $z = r$ we have $\arg z = 2\pi$, and hence $z^{1/2} = \sqrt{r}e^{\frac{1}{2}i(2\pi)} = \sqrt{r}e^{\pi i} = -\sqrt{r}$. So on this copy of the square root, the real numbers take their negative square roots.

What if we put the cut somewhere else? The job of the cut was to prevent us from circling around 0, but we didn't impose any other restrictions than that. We could have placed it along the positive real axis, or at an angle, say along the line $\{\lambda(1 + i) : \lambda \in \mathbb{R}\}$. It doesn't even need to be a straight line – a wiggly curve from 0 to $\infty$ would do.

All this would do is give us a particular value of the argument at each point, and hence allow us to define a function sensibly. If we want to use the right-hand half-plane for some reason, then we'll put the cut somewhere off in the left-hand half plane. And vice versa.

We mentioned $\infty$ above. That's worth pondering. When we cut the plane before to prevent us from circling 0, we couldn't stop cutting it. If we did stop somewhere, say $-1000$, then we could go for a long walk around the entire cut and back to where we started, and this would have increased the argument by $2\pi$, something we were trying to avoid. So we have to cut out to $\infty$.

But there is another reason for this: we mustn't circle $\infty$ either. Hmm, what does that even mean? Actually, let's think about the Earth.

The Antarctic Circle is a small circle about the South Pole, but we could think of it as a large circle about the North Pole. Imagine sliding the Antarctic Circle up the Earth, through the Tropic of Capricorn, until it gets to the Equator. It's still a circle around the South Pole, yes? But being the Equator, it must also be a circle around the North Pole. Indeed, keep sliding it north, throuth the Tropic of Cancer, to the Arctic Circle. So a circle about one Pole is also a circle about the other.

So we could think of our walks in $\mathbb{C}$ about 0 as being walks about $\infty$ as well. And if our walk "about the South Pole" causes some change in value, then thinking of the sphere there must be a point outside our route causing a change as well. And indeed, suppose we invert the plane/sphere, via $\zeta = 1/z$. We get $\log \zeta = -\log z$, and so the new origin $\zeta = 0$ has the same behaviour as $z = 0$ did before.

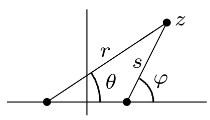
So in fact, our cut wasn't just starting at 0 and going out indefinitely "because otherwise we could circle 0". It was really joining together the two bad points 0 and $\infty$, so that we can no longer circle just one of them. Circling both is okay, as (thinking in terms of the sphere), that's the same as circling neither.

Let's finally (vaguely) define some actual terminology. A point $z_0$ is a *branch point* for a function $f$ if, when we travel on a small circle about $z_0$ and return to where we started, the value of $f$ at the end is different from the value of $f$ at the start. A *branch cut* is a section (usually a line) we delete from the plane in order to prevent us travelling about a branch point, rendering the function $f$ single-valued. A *branch* of the function $f$ is one of these resulting single-valued functions, with domain being $\mathbb{C} \setminus \{\text{branch cut}\}$.

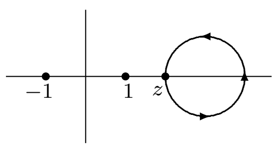
Let's consider another example.

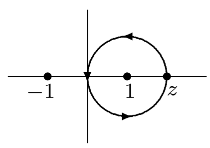
Let $f(z) = \sqrt{z^2 - 1} = (z+1)^{1/2}(z-1)^{1/2}$.

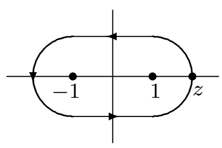
Write $z + 1 = re^{i\theta}$ and $z - 1 = se^{i\varphi}$, so then $f(z) = \sqrt{rs}\, e^{\frac{i}{2}(\theta+\varphi)}$.




Let us try doing some walks. In each case, we'll start at the point $z$ on the real axis and travel anticlockwise. Let's decide that $\arg z = 0$, and that the value of $f(z)$ is real and positive, i.e. that $\arg f(z) = 0$.




In the first picture, our walk doesn't go around either 1 or $-1$. The arguments $\theta$ and $\varphi$ both go down a bit at first, then back up to 0 as we return to the real axis, then both go up a bit. And when we finally return to $z$, both $\theta$ and $\varphi$ have returned to their starting values. Hence $f$ has also returned to its starting values, and there is no branching behaviour.

In the second picture, our walk goes around 1 but not $-1$. The argument $\theta$, as viewed from $-1$, goes up a bit, down a bit, then back up to where it started. However, the argument of $\varphi$, as viewed from 1, goes up and up, eventually increasing by $2\pi$ when we return to $z$. Since $\theta \to \theta$ and $\varphi \to \varphi + 2\pi$, we have $e^{\frac{i}{2}(\theta+\varphi)} \to e^{\frac{i}{2}(\theta+\varphi+2\pi)} = e^{\frac{i}{2}(\theta+\varphi)+i\pi} = -e^{\frac{i}{2}(\theta+\varphi)}$, and hence $f(z) \to -f(z)$. So 1 is a branch point of $f$.

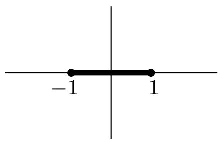
Similarly, if we were to walk around $-1$ but not 1, then we would get $f(z) \to -f(z)$, and so $-1$ is also a branch point of $f$.

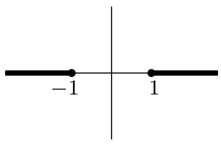
In the third picture, our walk goes around both 1 and $-1$. This time, both arguments $\theta$ and $\varphi$ go up and up, both eventually increasing by $2\pi$ when we return to $z$. Since $\theta \to \theta + 2\pi$ and $\varphi \to \varphi + 2\pi$, we have $e^{\frac{i}{2}(\theta+\varphi)} \to e^{\frac{i}{2}(\theta+\varphi+4\pi)} = e^{\frac{i}{2}(\theta+\varphi)+2i\pi} = e^{\frac{i}{2}(\theta+\varphi)}$, and hence $f(z) \to f(z)$. Indeed, this third picture is another example of "circling both is the same as circling neither".

This last picture tells us that, unlike $\log z$ earlier, the point $\infty$ isn't a branch point of this function. Indeed, informally, we see that for large $z$, we have $z+1 \approx z$ and $z-1 \approx z$, so $f(z) \approx \sqrt{z^2} = z$, which is single-valued. More formally, we let $\zeta = 1/z$ and get $\sqrt{1/\zeta^2 - 1} = \sqrt{1-\zeta^2}/\zeta$, which has no branch point at $\zeta = 0$.

Now that we know the behaviour, how do we cut the plane to prevent branching? We see that it's bad for us to circle just one of $\pm 1$, but that it's okay for us to circle both. This means that a cut which joins the two points together would be sufficient.

For example, we could cut along the real axis from $-1$ to 1, as shown in the first picture below. Or we could cut out the rest of the real line, as shown in the second picture below. Note that in the second picture, it doesn't mean that $\infty$ is also a branch point (we showed it wasn't). The cut is just happening to go through it, just like in the first picture the cut is just happening to go through 0.

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