$\tan^{-1}$

Suppose that $f(z)$ is an analytic function, and write $f(x+iy) = u(x,y) + iv(x,y)$. Then u and v satisfy the Cauchy-Riemann equations, and we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \Longrightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0$$

And similarly, $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$.

In other words, the real and imaginary parts of an analytic function are harmonic. More precisely, the functions u and v are harmonic at the point (x,y) if the function f is analytic at the point $z = x+iy$. From this, we see that a way to show that a given function $u(x,y)$ is harmonic is to find an analytic function $f(z)$ whose real or imaginary part is u.

For example, let $u(x,y) = e^x \cos y$. We notice that setting $v(x,y) = e^x \sin y$ gives $u + iv = e^z$. This is analytic for all z, and hence u and v are harmonic for all (x,y).

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Let $\psi(x,y) = \tan^{-1}(y/x)$. Is this harmonic? For $x \neq 0$ it isn't even defined, but elsewhere we have:

$$\left.\begin{aligned} \frac{\partial \psi}{\partial x} &= \frac{-y/x^2}{1+(y/x)^2} = \frac{-y}{x^2+y^2} \Longrightarrow \frac{\partial^2 \psi}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} \\ \frac{\partial \psi}{\partial y} &= \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2} \Longrightarrow \frac{\partial^2 \psi}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2} \end{aligned}\right\} \Longrightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

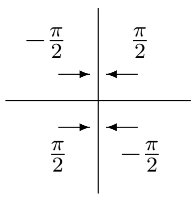
So, it's harmonic. Now, can we write ψ as the real or imaginary part of some analytic function f?

By solving/noticing, we might guess $\varphi(x,y) = \log\sqrt{x^2+y^2}$, because we're familiar with

$$\log z = \log|z| + i \arg z \ .$$

And since $\arg z = \tan^{-1}(y/x)$, we're done. Right? No! The two are often *not* equal.

Let's investigate $\tan^{-1}(y/x)$ a bit more closely. Suppose $y > 0$. For very small positive x, the value of y/x is large and positive, and so $\tan^{-1}(y/x)$ is just less than $\frac{\pi}{2}$. While for very small negative x, the value of y/x is large and negative, and so $\tan^{-1}(y/x)$ is just greater than $-\frac{\pi}{2}$. Similarly for $y < 0$.



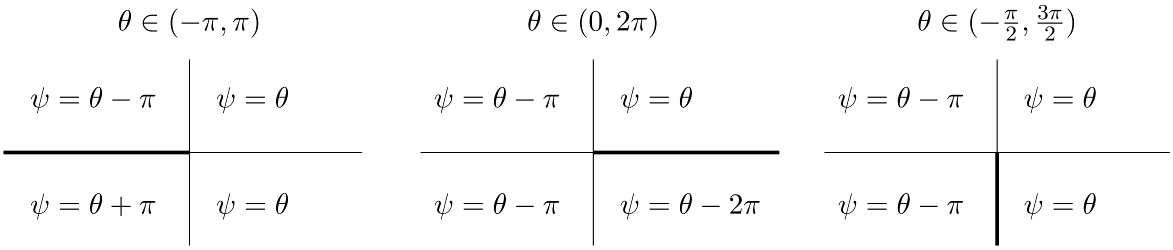
We see that this function isn't even continuous across the y-axis, so it certainly can't be harmonic there. It doesn't seem to equal the argument either.

What actually is the argument of z? Any non-zero point in $\mathbb{C}$ has an infinite choice of arguments, by adding 2π. E.g., the point $z = 1$ can be written as $e^{0\pi i}$, $e^{2\pi i}$, $e^{4\pi i}$, ...

But we don't want to use multiple names for points, so we select an argument for each point and stick with that. We could declare the range of $\arg z$ to be $(-\pi, \pi)$ or $(0, 2\pi)$ or $(-\frac{\pi}{2}, \frac{3\pi}{2})$, etc.

For these example ranges, how do $\psi = \tan^{-1}(y/x)$ and $\theta = \arg z$ compare?

The behaviour in each quadrant is as follows:



Suppose we decide to use the first of these. We define a branch of the logarithm by setting $\log z = \log|z| + i \arg z$, where $\arg z \in (-\pi, \pi)$, placing the branch cut along the negative real axis. Then we see that in the right-hand half-plane, we do indeed have $\psi = \mathrm{Im}(\log z)$, and so we have shown that $\tan^{-1}(y/x)$ is harmonic for $x > 0$.

What about $x < 0$? With our current definition of the logarithm, we can see that $\tan^{-1}(y/x)$ is harmonic in the top-left quadrant, where $x < 0$, $y > 0$, since $\psi = \mathrm{Im}(\log z - i\pi)$ there, and $\log z - i\pi$ is analytic there. And similarly in the bottom-left quadrant with $\log z + i\pi$. However, none of this works to show that $\tan^{-1}(y/x)$ is harmonic on the negative real axis, as our version of $\log z$ isn't even continuous there.

So instead, we could use the second picture above, and define a branch of logarithm by choosing $\arg z \in (0, 2\pi)$, placing the branch cut along the positive real axis. Then for all $x < 0$, we have $\psi = \mathrm{Im}(\log z - i\pi)$, and so it is harmonic there since $\log z - i\pi$ is analytic there.

However, we now see that it would probably have been more sensible to use the third picture, defining a branch of logarithm by choosing $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$, placing the branch cut along the negative imaginary axis. For with this cut we can do both halves: $\psi = \mathrm{Im}(\log z)$ for $x > 0$, and $\psi = \mathrm{Im}(\log z - i\pi)$ for $x < 0$. We have "hidden" the badness of $\log z$ along a line where we don't need it anyway, namely $x = 0$.

Conclusion: $\tan^{-1}(y/x)$ is harmonic. We can write it as the real or imaginary part of an analytic function. We just couldn't do it all at once.

What if we want to go the other way? That is, suppose we are given a range of $\theta = \arg z$, and we want to write it in terms of $\psi = \tan^{-1}(y/x)$. We can do this by adding suitable multiples of π to its natural value, and then defining it to be a suitable constant on the x-axis.

For example, suppose that we want our range of arguments to be $(-\pi, \pi)$. We have:

region	range of θ	range of ψ	relationship
$x < 0, y < 0$	$(-\pi, -\frac{\pi}{2})$	$(0, \frac{\pi}{2})$	$\theta = \psi - \pi$
$x = 0, y < 0$	$-\frac{\pi}{2}$	undefined	$\theta = -\frac{\pi}{2}$
$x > 0$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$\theta = \psi$
$x = 0, y > 0$	$\frac{\pi}{2}$	undefined	$\theta = \frac{\pi}{2}$
$x < 0, y > 0$	$(\frac{\pi}{2}, \pi)$	$(-\frac{\pi}{2}, 0)$	$\theta = \psi + \pi$

Alternatively, if we just want to find an expression which works in, say, the upper half-plane only, then rather than define it in three parts in terms of $\tan^{-1}(y/x)$, we could instead use a function like $\frac{\pi}{2} - \tan^{-1}(x/y)$. This is undefined on the x-axis, but for $y > 0$ takes values from 0 to π.

What about functions other than $\tan^{-1}(y/x)$?

Let $\varphi(x,y) = \log\sqrt{x^2+y^2}$. This function is clearly radially symmetric, and so if it's harmonic in one quadrant or half-plane then it must be harmonic in the whole plane (except at the origin, of course). We can show it's harmonic by writing it as $\mathrm{Re}(\log z)$, but we still need a branch cut for $\log z$.

We could first put the branch cut along the negative real axis, and on the cut plane we have $\varphi = \mathrm{Re}(\log z)$, which shows that φ is harmonic everywhere except the negative real axis. (Note: we haven't shown that it *isn't* harmonic on the negative real axis; we just haven't yet shown that it *is*.) So then we put the branch cut somewhere else, say along the positive real axis, and on the cut plane we have $\varphi = \mathrm{Re}(\log z)$, which shows that φ is harmonic everywhere except the positive real axis. Combining these, we see that φ is harmonic everywhere except the origin.

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Let's now consider $\psi(x,y) = \tan^{-1}\left(\dfrac{2x}{x^2+y^2-1}\right)$.

We really don't want to work out $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2}$ explicitly. But consider use $w(z)$ as given in the question. We find:

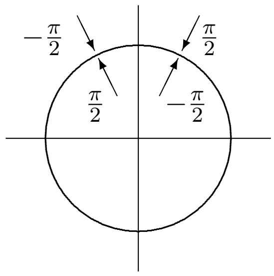
$$w(z) = \frac{z+i}{z-i} = \frac{x+iy+i}{x+iy-i} = \frac{(x+iy+i)(x-iy+i)}{(x+iy-i)(x-iy+i)} = \frac{x^2+y^2-1+2ix}{x^2+y^2+1-2y}$$

Then $\tan^{-1}\left(\dfrac{2x}{x^2+y^2-1}\right) = \tan^{-1}\left(\dfrac{\mathrm{Im}\, f(z)}{\mathrm{Re}\, f(z)}\right) = \arg w(z) = \mathrm{Im}\left(\log w(z)\right)$, maybe?

As before, not quite. We need to worry about what branch we're using for log, and what values of $\tan^{-1}$ we're expecting to get.

First, ψ isn't defined on the unit circle, but what happens as we approach the unit circle?

If (x,y) is in the right half-plane and just inside the unit circle, then $2x/(x^2+y^2-1)$ is large and negative, and so $u \approx -\frac{\pi}{2}$. While if (x,y) is in the right half-plane and just outside the unit circle, then $2x/(x^2+y^2-1)$ is large and positive, and so $\psi \approx \frac{\pi}{2}$. If (x,y) is in the left half-plane then we get a similar (but reversed) situation.

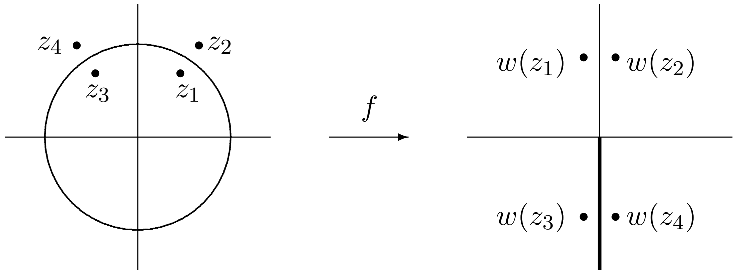


So we see that u has a jump of π across the unit circle, so we can't even define it a value and hope to get continuity. So it's certainly not harmonic on the unit circle.

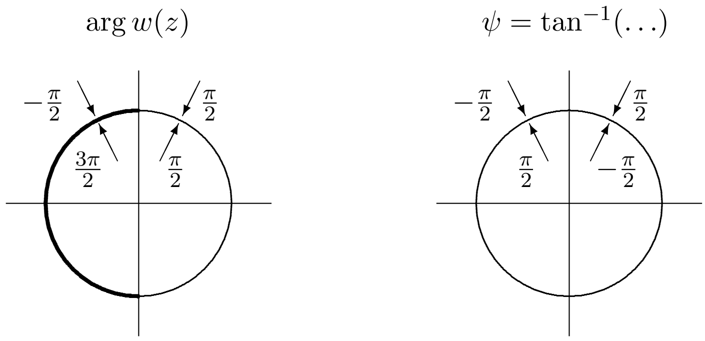
Recalling our experience with $\tan^{-1}(y/x)$ earlier, we could deal with the inside and outside of the disc separately, by choosing a branch of log that worked in two different regions. However, like with $\tan^{-1}(y/x)$, it would be nice if we could hide the badness of log where we don't need to use it anyway, namely the unit circle.

So, where does $w(z)$ send the unit circle? Since $w(-i) = 0$, $w(1) = i$, $w(i) = \infty$ and $w(0) = -1$, we see that the unit circle maps to the imaginary axis, and that the unit disc maps to the left half-plane. Let us therefore place the branch cut on the imaginary axis, by choosing arguments to be in $(-\frac{\pi}{2}, \frac{3\pi}{2})$.

We get the following effects.



From this, we see the difference in behaviour of $\arg w(z)$ and ψ.



So, we can now finally show that ψ is harmonic except on the unit circle.

On our cut plane, $\log z$ is analytic and so $\arg z = \text{Im}(\log z)$ is harmonic. Now, for $|z| > 1$, we have $w(z)$ in the right half-plane, and so, with our chosen cut, $\log w(z)$ is analytic. And for $|z| < 1$, we have $w(z)$ in the left half-plane, and so again $\log w(z)$ is analytic. So for all $|z| \neq 1$, we have $\log w(z)$ analytic and hence $\arg w(z)$ is harmonic.

Outside the unit circle, we have $u = \arg w(z)$, and hence u is harmonic there. Inside the unit circle, we have $u = \arg w(z) - \pi$, and hence ψ is harmonic there also.